14-1100 Sep 18, hour 6: Jordan-Holder

September-17-14 2:07 PM

Read Along. Pavel Etingof's "Groups Around Us", Lang's page 57.

Riddle Along. Your turn!

Today's Menu. Jordan-Hölder.

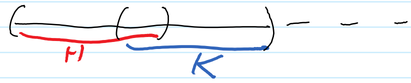
Reminders. $\phi: G \to H$:

$$G/\ker\phi \cong \operatorname{im}\phi$$

$$\dim V - \operatorname{nullity} L = \operatorname{rank} L$$

$H < N_G(K)$:

$$HK/K \cong H/H\cap K$$



$$\frac{G/K}{H/K} \cong G/H$$

$G \trianglerighteq N$: $\left\{\begin{matrix}\text{subgroups}\\ \text{of } G/N\end{matrix}\right\} \leftrightarrow \{H : N < H < G\}$

on board

Definition A simple group.

"A prime", (in fact, $\mathbb{Z}/n$ is simple iff n is a prime)

yet $S_3 \triangleright A_3 = \langle(123)\rangle = \mathbb{Z}_3$; $S_3/A_3 = \mathbb{Z}/2$

$\mathbb{Z}/6 \triangleright 2\mathbb{Z}/6 = \{0,2,4\} = \mathbb{Z}/3$; $\mathbb{Z}/6 / 2\mathbb{Z}/6 = \mathbb{Z}/2$

The Jordan-Hölder Theorem. Let G be a finite group. Then there exist a sequence

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \dots \triangleright G_n = \{e\} \quad \text{s.t. } H_i = G_i/G_{i-1}$$

is simple. Furthermore, the sequence (H_i), the "composition series" of G, is unique up to a permutation.

$4 \to A_4 \to A_3$
$12 \quad 3$

Example $\underset{24}{S_4} \triangleright \underset{12}{A_4} \triangleright \underset{4}{\begin{matrix}(12)(34)\\(13)(24)\\(14)(23)\end{matrix}} \triangleright \underset{2}{(12)(34)} \triangleright \{e\}$

Proof by induction on $|G|$.

Existance: Let G_1 be a maximal normal proper subgroup.

Uniqueness: use the "diamond principle":

$$\begin{tikzcd}
 & G \arrow[dl, dash, "H_1"'] \arrow[dr, dash, "H_1'"] & \\
G_1 \arrow[dr, dash, "H_1'"'] & & G_1' \arrow[dl, dash, "H_1"] \\
 & G_1 \cap G_1' &
\end{tikzcd}$$

$G \rhd G_1 \rhd G_2 \dots$

$G \rhd G_1' \rhd G_2' \dots$

<u>Claim</u> $G = G_1 G_1'$

<u>Pf</u> $G_1 G_1'$ is normal in G yet bigger that G_1, G_1'.

done line

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sign: $S_n \to \{\pm 1\}$ by $\operatorname{sign}(\sigma) = (-1)^\sigma = \operatorname{sign}\left(\prod_{i<j}(\sigma i - \sigma j)\right)$

$$= \prod_{\{i,j\} \subset \{1, \dots n\}} s_{i,j}(\sigma) \qquad s_{i,j}(\sigma) = \operatorname{sign}\left(\frac{\sigma i - \sigma j}{i-j}\right)$$

$$(-1)^{\sigma\tau} \overset{?}{=} \operatorname{Sign}\left[\prod_{\{i,j\}} \frac{\sigma\tau i - \sigma\tau j}{\tau i - \tau j} \; \frac{\tau i - \tau j}{i-j}\right] = (-1)^\sigma (-1)^\tau$$

Every permutation is a product of transpositions, the parity is the parity of the number of transpositions.

Theorem. A_n is simple for $n \neq 4$. [Proof as in Lang's]

Cycle Decomposition. $(12)(345) = [21453] = 21453$

<u>Claim</u> If $\sigma = (a_1 \dots a_k)$ and $\tau = [\tau_1 \tau_2 \dots \tau_n]$, then

$$\sigma^\tau = \tau^{-1} \sigma \tau = (\tau^{-1}(a_1), \tau^{-1} a_2, \dots)$$

<u>Corollary</u> σ is conjugate to σ' iff they have the same cycle lengths

<u>Corollary</u> $\#$(Conjugacy classes of S_n) $= p(n)$

Lemma 1. Every element of A_n is a product of 3-cycles.

Pf $(12)(23)=(123)$, $(123)(234)=(12)(34)$ ...

Lemma 2. If $N \triangleleft A_n$ contains a 3-cycle, then $N=A_n$

Pf WLOG, $(123)\in N$. Claim For $\sigma\in S_n$, $(123)^\sigma \in N$ $\left(\begin{matrix}\sigma\in A_n \checkmark\\ \sigma=(12)\sigma' \checkmark\end{matrix}\right)$

So N contains all 3-cycles ... $\square$

Now take $N \triangleleft A_n$ w/ $N\neq\{1\}$

Case 1. N contains an element w/ cycle of length ≥ 4

$\sigma=(123456)\sigma'\in N$ $\quad \sigma^{-1}(123)\sigma(123)^{-1}=(136)$

Case 2. N contains an element $\sigma=(123)(456)\sigma'$

Consider $\sigma^{-1}(124)\sigma(124)^{-1}=(14263)$

Case 3. N contains $\sigma=(123)$ (product of pairs)

Then $\sigma^2=(132)$ ...

Case 4. Every element of N is a product of disjoint 2-cycles

$\sigma=(12)(34)\sigma' \Rightarrow \sigma^{-1}(123)\sigma(123)^{-1}=(13)(24)=\tau\in N$

$\Rightarrow \tau^{-1}(125)\tau(125)^{-1}=(13452)\in N$