14-1100 Nov 27, hour 32: The "ring" of modules

November-26-14 3:39 PM

Return HW4!

Course evals: 2/17. Vote and warn others!

<u>Definition</u> A "tensor product" $M\otimes N$ is a module $M\otimes N$ along with a bilinear $\tau: M\times N \longrightarrow M\otimes N$ s.t.

$$\begin{tikzcd} M\times N \arrow[r, "\tau", "\text{bilinear}"'] \arrow[dr, "\rho"', "\text{bilinear}"] & M\otimes N \arrow[d, dashed, "\exists!\ \text{linear}"] \\ & P \end{tikzcd}$$

<u>Thm</u> $M\otimes N$ exists ✓ & is unique up to isomorphism.
today

Example. $\dim V\otimes W = (\dim V)\cdot(\dim W)$

Proof of uniqueness.

Example. If $q = \gcd(a,b)$, $q = sa+tb$, $\frac{R}{\langle a\rangle}\otimes\frac{R}{\langle b\rangle} \cong \frac{R}{\langle q\rangle}$

Pf. $[r_1]_a\otimes[r_2]_b \longrightarrow [r_1\cdot r_2]_q$ $\qquad [q]\otimes[1] = [sa+tb]\otimes[1] = 0$

$[r]_q \longrightarrow [r]_a\otimes[1]_b$ $\qquad [r_1 r_2]\otimes[1] = [r_1][r_2]$

Example. $\tau: F(X)\otimes F(Y) \to F(X\times Y)$

1. Always injective! [not so easy!]
2. Isomorphism if X or Y are finite.
3. Not surjective if $R=\mathbb{Z}$, X, Y are infinite. [not at all obvious!]

Theorem. $(R\text{-mod}, \oplus, \otimes, 0, R)$ is a "ring".

Theorem. $(M,N) \longmapsto M\otimes N$ is a "bifunctor".

done line

Example. $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}^n \cong \mathbb{Q}^n$ "Extension of scalars".
a $\mathbb{Q}$-module

line

Example. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$ ← a $\mathbb{Q}$-module! "Extension of scalars".

In general, given $\phi: R \to S$ a ring morphism, S is an R module & set $M_S := S \otimes_R M$. Then M_S is an S-module and $R^n_S = S^n$.

Prop. For any domain R there is a unique field $Q(R)$ s.t.

$$\begin{array}{ccc} R & \xrightarrow{1-1} & Q(R) \\ & \searrow & \downarrow \exists! \\ & & F \end{array}$$

"The field of fractions"

Proof later.

<u>Claim</u> If M is torsion $\left[\begin{array}{c}\forall m \in M\ \exists r \in R \setminus \{0\} \\ rm = 0\end{array}\right]$ Then $M_{Q(R)} = 0$.

$a \otimes m = r\left(\frac{a}{r} \otimes m\right) = \frac{a}{r} \otimes rm = 0$

<u>Prop</u> If $M \cong R^K \oplus \bigoplus R/\langle p_i^{s_i} \rangle$, then

1. $\dim_{Q(R)} M_{Q(R)} = K$
2. $\dim_{R/\langle p \rangle} M_{R/\langle p \rangle} = K + |\{i : p_i \sim p\}|$
3. $\dim_{R/\langle p \rangle} \operatorname{im}(m \mapsto p^s m)_{R/\langle p \rangle} = K + |\{i : p_i \sim p \ \& \ s < s_i\}|$

$R/\langle p \rangle$ is a field because in a PID $\langle p \rangle$ is maximal

as

$$\operatorname{im}(m \mapsto p^s m) \cong \begin{cases} p^s R \cong R & \text{on } R \\ R/\langle q^t \rangle & \text{on } R/\langle q^t \rangle \quad q \nsim p \\ 0 & \text{on } R/\langle p^t \rangle \quad s \geq t \\ R/\langle p^{t-s} \rangle & \text{on } R/\langle p^t \rangle \quad s < t \end{cases}$$

and

$$\ker(m \mapsto p^s m) \cong \begin{cases} 0 & \text{on } R \\ 0 & \text{on } R/\langle q^t \rangle \quad q \nsim p \\ R/\langle p^t \rangle & \text{on } R/\langle p^t \rangle \quad s \geq t \\ R/\langle p^s \rangle & \text{on } R/\langle p^t \rangle \quad s < t \end{cases}$$

$R/\langle p^s \rangle \hookrightarrow \ker$ by $[r]_{p^s} \mapsto [p^{t-s} r]_{p^t}$

So such a decomposition is unique!

Localization & Fields of Fractions. Let R be a commutative domain

<u>Def</u> A multiplicative subset S of $R \setminus \{0\}$. (contains 1, closed under $\times$)

<u>Examples</u> $R \setminus \{0\}$, $R \setminus P$ (P prime), Powers of $a \neq 0$.

Definition $S^{-1}R = \{\frac{r}{s}\}/$

$\frac{r_1}{s_1} \sim \frac{r_2}{s_2}$ if $r_1 s_2 = r_2 s_1$

$\left[\frac{r_1}{s_1} \sim \frac{r_2}{s_2}, \frac{r_2}{s_2} \sim \frac{r_3}{s_3} \Rightarrow r_1 s_2 = r_2 s_1, r_2 s_3 = r_3 s_2 \Rightarrow \right.$

$\left. r_1 s_2 s_3 = r_2 s_1 s_3 = s_1 r_3 s_2 \Rightarrow r_1 s_3 = r_3 s_1 \right]$

$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \ldots$

$\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \ldots$

$R \setminus \{0\}$ — "Field of Fractions $Q(R)$"

$R \setminus P$ — "localization at P"

$\{2^n\}$ — "dyadic rationals".

$R \to S^{-1}R$ is injective