14-1100 Nov 13, hour 28: Main Theorem - Existence

November-12-14 12:53 PM

Goal: M f.g. / PID $R \Rightarrow$

$$M = R^k \oplus \bigoplus_{i=1}^{n} R/\langle p_i^{s_i}\rangle \qquad p_i \text{ prime},\ s_i \in \mathbb{Z}_{>0}$$

There is a map from $n\times m$ matrices to f.g. modules:

$$A \longmapsto \quad R^m \xrightarrow{A} R^n \longrightarrow R^n/\mathrm{im}A =: M_A$$

Equally well, $n\times X$ matrices to f.g. modules:

$$A \longmapsto \quad R^X \xrightarrow{A} R^n \longrightarrow R^n/\mathrm{im}A =: M_A$$

$M_{n\times X}(R) \longrightarrow$ f.g. modules is surjective.

Examples. (1), (a), (0)

Exercise. If $C = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B\end{array}\right)$, then $M_C = M_A \oplus M_B$

Comment. May add/remove 0 columns.

$$\begin{array}{ccc} R^X & \xrightarrow{A} & R^g \\ \uparrow Q & \circlearrowleft & \downarrow P \\ R^X & \xrightarrow{A'} & R^g \end{array}$$

Claim if P,Q are invertible on the left, then $M = R^g/\mathrm{im}A$ and $M' = R^g/\mathrm{im}A'$ are isomorphic.

Pf $\Phi: M \to M'$ by $[\alpha]_{\mathrm{im}A} \longrightarrow [P\alpha]_{\mathrm{im}A'}$

P can be interpreted as $g\times g$ matrix

Q can be interpreted as an $X\times X$ column-finite matrix; $A' = PAQ$

.... Can do arbitrary invertible row operations on A,

and arbitrary invertible column ops, provided each column is touched finitely many times.

Of all the matrices reachable from A, let A'

be the one having an entry with the smallest D-H norm; wlog, that entry is a_{11}.

Claim a_{11} divides all other entries in its row & column.

Pf 1 for a Euclidean domain.

Pf 2 In a PID, if $q = \gcd(a,b) = sa+tb$, then

$$(a\ b)\begin{pmatrix} s & -b/q \\ t & a/q \end{pmatrix} = (q\ 0), \text{ while } \begin{pmatrix} s & -b/q \\ t & a/q \end{pmatrix}^{-1} = \begin{pmatrix} a/q & b/q \\ -t & s \end{pmatrix} \quad \square$$

$\Longrightarrow$ W.l.o.g, the row & column of a_{11} are 0 (except for a_{11})

$\Longrightarrow$ all entries of A are divisible by a_{11}:

$$A = \begin{pmatrix} a_{11} & \text{---}\ 0\ \text{---} \\ 0 & \\ \vdots & A_1 \ \text{all entries divisible by } a_{11} \\ 0 & \end{pmatrix}$$

Continue to get $A \sim \left(\begin{array}{cc|c} a_{11} & & \\ & a_{22} & 0 \\ \hline & 0 & 0 \end{array}\right)$ (w.l.o.g., A is square)

So $M \cong \bigoplus_{i=1}^{g} R/\langle a_{ii}\rangle \cong R^k \oplus \bigoplus R/\langle a_i\rangle$

$a_1 \mid a_2 \mid \dots \mid a_n$