

next class: Wed 1-3 OH 3-4.

Course evals: 4/17 Vote and warn others!

Goal. Uniqueness in the structure thm.

Theorem. $(R\text{-mod}, \oplus, \otimes, 0, R)$ is a "ring".

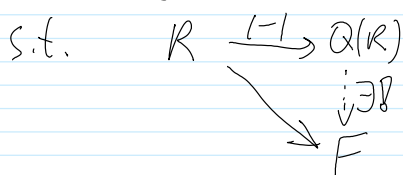
Theorem. $(M, N) \mapsto M \otimes N$ is a "bifunctor".

start line

Example. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$ "Extension of scalars".
 \leftarrow a \mathbb{Q} -module

In general, given $\phi: R \rightarrow S$ a ring morphism, S is an R module & set $M_S := S \otimes_R M$. Then M_S is an S -module and $R_S^n = S^n$.

Prop. For any domain R there is a unique field $\mathbb{Q}(R)$



"the field of fractions"
proof: later.

Claim IF M is torsion $\left[\forall m \in M \exists r \in R \setminus \{0\} \text{ s.t. } rm = 0 \right]$ then $M_{\mathbb{Q}(R)} = 0$.

$$a \otimes m = r \left(\frac{a}{r} \otimes m \right) = \frac{a}{r} \otimes rm = 0$$

Prop IF $M \cong R^k \oplus \bigoplus R/\langle p_i, s_i \rangle$, then

1. $\dim_{\mathbb{Q}(R)} M_{\mathbb{Q}(R)} = k$

2. $\dim_{R/\langle p \rangle} M_{R/\langle p \rangle} = k + |\{i : p_i \sim p\}|$

3. $\dim_{R/\langle p \rangle} \text{im}(M \rightarrow p^s M)_{R/\langle p \rangle} = k + |\{i : p_i \sim p \ \& \ s < s_i\}|$

$R/\langle p \rangle$ is a field because in a PID $\langle p \rangle$ is maximal

$$\text{im}(M \rightarrow p^s M) \cong \begin{cases} p^s R \cong R & \text{on } R \\ R/\langle q^t \rangle & \text{on } R/\langle q^t \rangle \quad q \neq p \\ 0 & \text{on } R/\langle p^t \rangle \quad s \geq t \\ R/\langle p^t s \rangle & \text{on } R/\langle p^t \rangle \quad s < t \end{cases}$$

and

$$\ker(M \rightarrow p^s M) \cong \begin{cases} 0 & \text{on } R \\ 0 & \text{on } R/\langle q^t \rangle \quad q \neq p \\ R/\langle p^t \rangle & \text{on } R/\langle p^t \rangle \quad s \geq t \\ R/\langle p^s \rangle & \text{on } R/\langle p^t \rangle, \quad s < t \end{cases}$$

$$\left\{ \begin{array}{l} \text{on } R/\langle p^t \rangle \supseteq t \\ R/\langle p^{t+s} \rangle \text{ on } R/\langle p^t \rangle \supseteq t \end{array} \right.$$

$$\left\{ \begin{array}{l} R/\langle p^t \rangle \text{ on } R/\langle p^{t-1} \rangle \supseteq t \\ R/\langle p^s \rangle \text{ on } R/\langle p^t \rangle \supseteq t \\ R/\langle p^s \rangle \mapsto \ker \text{ by } [r]_{p^s} \mapsto [p^{t-s}r]_{p^t} \end{array} \right.$$

So such a decomposition is unique!

Localization & Fields of fractions. Let R be a commutative

Def A multiplicative subset S of $R \setminus \{0\}$. (contains 1, closed under \times)

Examples $R \setminus \{0\}$, $R \setminus P$ (P prime), Powers of $a \neq 0$.

Definition $S^{-1}R = \left\{ \frac{r}{s} \right\} / \frac{r_1}{s_1} \sim \frac{r_2}{s_2} \text{ if } r_1 s_2 = r_2 s_1$

$$\left[\frac{r_1}{s_1} \sim \frac{r_2}{s_2}, \frac{r_2}{s_2} \sim \frac{r_3}{s_3} \Rightarrow r_1 s_2 = r_2 s_1, r_2 s_3 = r_3 s_2 \Rightarrow \right. \quad \left. \frac{r_1}{s_1} + \frac{r_2}{s_2} = \dots$$

$$\left. r_1 s_2 s_3 = r_2 s_1 s_3 = s_1 r_3 s_2 \Rightarrow r_1 s_3 = r_3 s_1 \right] \quad \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \dots$$

$R \setminus \{0\}$ - "Field of fractions $\mathbb{Q}(R)$ "

$R \setminus P$ - "localization at P "

$\{2^n\}$ - "dyadic rationals".

$R \rightarrow S^{-1}R$
is injective

all done