14-1100 Dec 1, hours 33-34: Uniqueness in the structure theorem

November-29-14 7:29 PM

Next class: Wed 1-3 OH 3-4.

Course evals: 4/17 Vote and warn others!

Goal. Uniqueness in the structure thm.

theorem. $(R\text{-mod}, \oplus, \otimes, 0, R)$ is a "ring".

theorem. $(M,N) \longmapsto M \otimes N$ is a "bifunctor".

start line

Example. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$ ← a $\mathbb{Q}$-module! "Extension of scalars".

In general, given $\phi: R \to S$ a ring morphism, S is an R module & set $M_S := S \otimes_R M$. Then M_S is an S-module and $R^n_S = S^n$.

Prop. For any domain R there is a unique field $Q(R)$ "the field of fractions"

s.t.
$$\begin{array}{ccc} R & \xrightarrow{1-1} & Q(R) \\ & \searrow & \downarrow \exists! \\ & & F \end{array}$$

proof: later.

Claim If M is torsion $\left[\forall m \in M\ \exists r \in R \setminus \{0\}\ rm = 0\right]$ Then $M_{Q(R)} = 0$.

$a \otimes m = r\left(\frac{a}{r} \otimes m\right) = \frac{a}{r} \otimes rm = 0$

Prop If $M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i} \rangle$, Then

1. $\dim_{Q(R)} M_{Q(R)} = k$
2. $\dim_{R/\langle p \rangle} M_{R/\langle p \rangle} = k + |\{i : p_i \sim p\}|$
3. $\dim_{R/\langle p \rangle} \operatorname{im}(m \mapsto p^s m)_{R/\langle p \rangle} = k + |\{i : p_i \sim p \ \&\ s < s_i\}|$

$R/\langle p \rangle$ is a field because in a PID $\langle p \rangle$ is maximal

as
$$\operatorname{im}(m \mapsto p^s m) \cong \begin{cases} p^s R \cong R & \text{on } R \\ R/\langle q^t \rangle & \text{on } R/\langle q^t \rangle \quad q \not\sim p \\ 0 & \text{on } R/\langle p^t \rangle \quad s \geq t \\ R/\langle p^{t-s} \rangle & \text{on } R/\langle p^t \rangle \quad s < t \end{cases}$$

and
$$\ker(m \mapsto p^s m) \cong \begin{cases} 0 & \text{on } R \\ 0 & \text{on } R/\langle q^t \rangle \quad q \not\sim p \\ R/\langle p^t \rangle & \text{on } R/\langle p^t \rangle \quad s \geq t \\ R/\langle p^s \rangle & \text{on } R/\langle p^t \rangle, \ s < t \end{cases}$$

$\quad$ on $R/\langle p^t\rangle$ $\quad s \geq t$

$R/\langle p^{t-s}\rangle$ on $R/\langle p^t\rangle$ $\quad s < t$

$R/\langle p^t\rangle$ on $R/\langle p^t\rangle$ $\quad s \geq t$

$R/\langle p^s\rangle$ on $R/\langle p^t\rangle$ $\quad s < t$

$R/\langle p^s\rangle \mapsto \ker$ by $[r]_{p^s} \mapsto [p^{t-s} r]_{p^t}$

So such a decomposition is unique!

Localization & Fields of Fractions.

Let R be a commutative domain

Def A multiplicative subset S of $R \setminus \{0\}$. (contains 1, closed under $\times$)

Examples $R\setminus\{0\}$, $R \setminus P$ (P prime), Powers of $a \neq 0$.

Definition $S^{-1}R = \left\{\frac{r}{s}\right\} \Big/ \frac{r_1}{s_1} \sim \frac{r_2}{s_2}$ if $r_1 s_2 = r_2 s_1$

$\left[\frac{r_1}{s_1} \sim \frac{r_2}{s_2}, \frac{r_2}{s_2} \sim \frac{r_3}{s_3} \Rightarrow r_1 s_2 = r_2 s_1, r_2 s_3 = r_3 s_2 \Rightarrow \right.$

$\left. r_1 s_2 s_3 = r_2 s_1 s_3 = s_1 r_3 s_2 \Rightarrow r_1 s_3 = r_3 s_1 \right]$

$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \ldots$

$\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \ldots$

$R \setminus \{0\}$ — "Field of Fractions $Q(R)$"

$R \setminus P$ — "localization at P"

$\{2^n\}$ — "dyadic rationals".

$R \to S^{-1}R$ is injective

all done