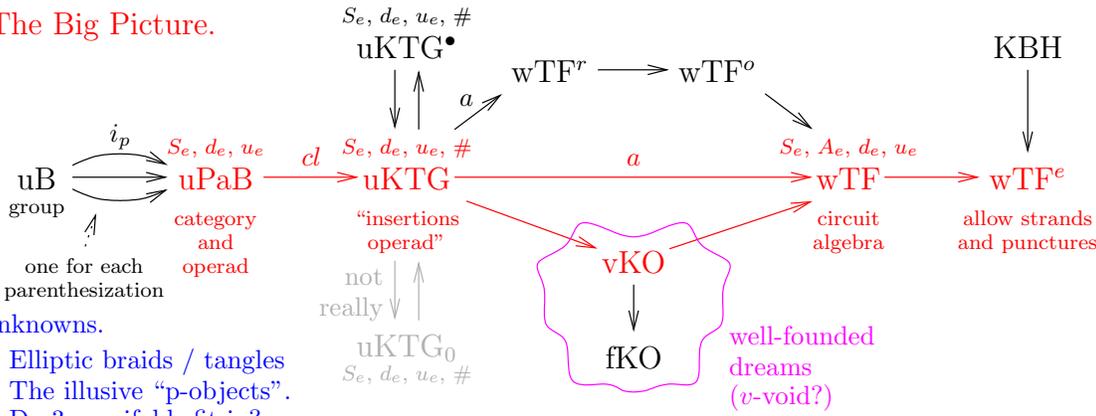


The Big Picture.



Unknowns.

1. Elliptic braids / tangles
2. The illusive "p-objects".
3. Do 3-manifolds fit in?

The  $u$ -Universe.  $uKTG = PA \langle uGens | uRels | uOps \rangle_{1,1}$

The  $w$ -World.  $wTF = CA \langle wGens | wRels | uOps \rangle$

**uJ**

xings.  $\nearrow + \searrow -$

vertices  $\begin{matrix} \nearrow + \\ \searrow - \end{matrix}$

Framing, etc. strands are framed

OPS

- unzip
- on switch  $S$
- delete

**wJ**

$R_{12}, R_{21}, V_{12}, V_{21}$

"thin" strands are oriented up

+CAP

• strands are ribbons w/ two sides + framed (two sides mirror)

• CA

• unzip (use framing)

• antipode  $A$

• delete • cap

Wen  $\begin{matrix} \nearrow \\ \searrow \end{matrix} \neq \begin{matrix} \searrow \\ \nearrow \end{matrix}$  fw

$W^2 = 1$

"switch"  $S = WAW$

Map a  $uJ \rightarrow wJ$

$\nearrow \mapsto R_{12}$   $\searrow \mapsto R_{21}$

$\nearrow \mapsto V_{12}$   $\searrow \mapsto V_{21}$

band comes from BB framing

framing comes from  $u$ -framing

$wJ \xrightarrow{u, S, d} uJ$

$wJ \xrightarrow{u, A, d} wJ$

Map  $\alpha \mathcal{A}^u \rightarrow \mathcal{A}^w$

$H \mapsto H + H$

Compatibility:

$uJ \xrightarrow{Z^u} \mathcal{A}^u$

$wJ \xrightarrow{Z^w} \mathcal{A}^w$

Theta

$\Theta = \alpha Z^u \left( \begin{matrix} \nearrow \\ \searrow \end{matrix} \right)$

**Z<sup>u</sup>**

$\nearrow \mapsto R_u = e^{\frac{\pi}{2}}$

$\searrow \mapsto R_u = e^{-\frac{\pi}{2}}$

$(R_u^{12} = R_u^{21})$

$\nearrow \mapsto \begin{matrix} \square \\ \square \end{matrix}$

$\searrow \mapsto \begin{matrix} \square \\ \square \end{matrix}$

(adjustment cancels for balanced diagrams)

**Z<sup>w</sup>**

$\nearrow \mapsto R_{12} = e^{\frac{\pi}{2}}$  etc.

$\searrow \mapsto \begin{matrix} \square \\ \square \end{matrix}$  etc

$\cap \mapsto \square$

③ Unitarity:  $V \cdot A \cdot A_2(V) = 1$

$A_1 A_2 V = \begin{matrix} \nearrow \\ \searrow \end{matrix} = \begin{matrix} \searrow \\ \nearrow \end{matrix} = 1$

④ Vertical flip  $V(S, S_2 V) = R$

$S S_2 V = \begin{matrix} \nearrow \\ \searrow \end{matrix} = \begin{matrix} \searrow \\ \nearrow \end{matrix} = R$

⑤ Cap.  $c_{f_2}(V C^{(1,2)}) = c_{f_2}(C C^{\otimes 2})$

⑥ Sides nondeg.  $d_1 V = d_2 V = 1$

Overhand rule?

Equations

① R4.  $R^2 R^{13} V = V R^{(1,2)3}$

$\begin{matrix} \nearrow \\ \searrow \end{matrix} = \begin{matrix} \searrow \\ \nearrow \end{matrix}$

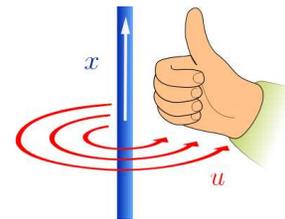
② Twist  $V\Theta = R V^{21}$

$\Theta = V^{-1} R V^{21}$

$V^{-1} = \begin{matrix} \searrow \\ \nearrow \end{matrix}$

$A_1 A_2 V = \begin{matrix} \nearrow \\ \searrow \end{matrix} = 1$

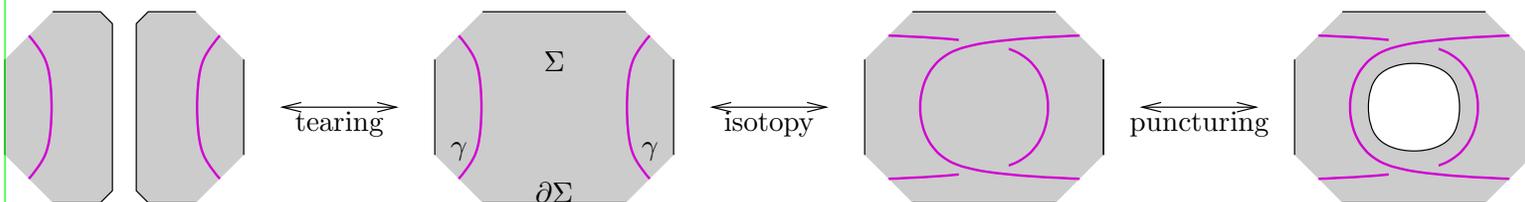
If  $x$  is an oriented  $S^1$  and  $u$  is an oriented  $S^2$  in an oriented  $S^4$  (or  $\mathbb{R}^4$ ) and the two are disjoint, their linking number  $l_{ux}$  is defined as follows. Pick a ball  $B$  whose oriented boundary is  $u$  (using the “outward pointing normal” convention for orienting boundaries), and which intersects  $x$  in finitely many transversal intersection points  $p_i$ . At any of these intersection points  $p_i$ , the concatenation of the orientation of  $B$  at  $p_i$  (thought of a basis to the tangent space of  $B$  at  $p_i$ ) with the tangent to  $x$  at  $p_i$  is a basis of the tangent space of  $S^4$  at  $p_i$ , and as such it may either be positively oriented or negatively oriented. Define  $\sigma(p_i) = +1$  in the former case and  $\sigma(p_i) = -1$  in the latter case. Finally, let  $l_{ux} := \sum_i \sigma(p_i)$ . It is a standard fact that  $l_{ux}$  is an isotopy invariant of  $(u, x)$ .



An efficient thumb rule for deciding the linking-number signs for a balloon  $u$  and a hoop  $x$  presented using our standard notation is the “right-hand rule” of the figure on the right, shown here without further explanation. The lovely figure is adopted from [Wikipedia: Right-hand rule].

$v$ -Knots are oriented knots drawn on an oriented surface  $\Sigma$  (meaning, “embedded in  $\Sigma \times [-\epsilon, \epsilon]$ ”), modulo “stabilization”, which is the addition and/or removal of empty handles (handles that do not intersect with the knot). We prefer an equivalent, yet even more bare-bones approach. For us, a virtual knot is an oriented knot  $\gamma$  drawn on a “virtual surface  $\Sigma$  for  $\gamma$ ”. More precisely,  $\Sigma$  is an oriented surface that may have a boundary,  $\gamma$  is drawn on  $\Sigma$ , and the pair  $(\Sigma, \gamma)$  is taken modulo the following relations:

- Isotopies of  $\gamma$  on  $\Sigma$  (meaning, in  $\Sigma \times [-\epsilon, \epsilon]$ ).
- Tearing and puncturing parts of  $\Sigma$  away from  $\gamma$ :



(We call  $\Sigma$  a “virtual surface” because tearing and puncturing imply that we only care about it in the immediate vicinity of  $\gamma$ ).

We can now define a map  $\delta_0$ , defined on  $v$ -knots and taking values in ribbon tori in  $\mathbb{R}^4$ : given  $(\Sigma, \gamma)$ , embed  $\Sigma$  arbitrarily in  $\mathbb{R}^3_{xyz} \subset \mathbb{R}^4$ . Note that the unit normal bundle of  $\Sigma$  in  $\mathbb{R}^4$  is a trivial circle bundle and it has a distinguished trivialization, constructed using its positive- $t$ -direction section and the orientation that gives each fiber a linking number  $+1$  with the base  $\Sigma$ . We say that a normal vector to  $\Sigma$  in  $\mathbb{R}^4$  is “near unit” if its norm is between  $1 - \epsilon$  and  $1 + \epsilon$ . The near-unit normal bundle of  $\Sigma$  has as fiber an annulus that can be identified with  $[-\epsilon, \epsilon] \times S^1$  (identifying the radial direction  $[1 - \epsilon, 1 + \epsilon]$  with  $[-\epsilon, \epsilon]$  in an orientation-preserving manner), and hence the near-unit normal bundle of  $\Sigma$  defines an embedding of  $\Sigma \times [-\epsilon, \epsilon] \times S^1$  into  $\mathbb{R}^4$ . On the other hand,  $\gamma$  is embedded in  $\Sigma \times [-\epsilon, \epsilon]$  so  $\gamma \times S^1$  is embedded in  $\Sigma \times [-\epsilon, \epsilon] \times S^1$ , and we can let  $\delta_0(\Sigma, \gamma)$  be the composition

$$\gamma \times S^1 \hookrightarrow \Sigma \times [-\epsilon, \epsilon] \times S^1 \hookrightarrow \mathbb{R}^4,$$

which is a torus in  $\mathbb{R}^4$ , oriented using the given orientation of  $\gamma$  and the standard orientation of  $S^1$ .

