Alekseev-Torossian statement. There is an element $F \in \text{TAut}_2$ with
\[
F(x + y) = \log e^x e^y
\]
and $j(F) \in \im \delta \subset \text{tr}_2$, where for $a \in \text{tr}_1$, 
\[
\delta(a) := a(x) + a(y) - a(\log e^x e^y).
\]

Free Lie statement (Kashiwara-Vergne). There exist convergent Lie series $F$ and $G$ so that with $z = \log e^x e^y$,
\[
x + y - \log e^x e^y = (1 - e^{-\text{ad} x}) F + (e^{\text{ad} y} - 1)G
\]
\[
\text{tr}(\alpha x) = \frac{1}{2} \text{tr} \left( \frac{e^{-\text{ad} x}}{e^{\text{ad} y} - 1} \right)
\]

Wheels and Trees. With $P$ for Primitives,
\[
0 \to (\text{wheels}) \to \text{PA}^w(\uparrow_n) \to (\text{trees}) \to 0
\]
with as $\text{proj} \mathcal{K}^w(\uparrow_n) \cong \mathcal{U}((\text{trees}) \times (\text{wheels}))$.

Some A-T Notions. $a_n$ is the vector space with bases $x_1, \ldots, x_n$, $\text{Lie}(a_n)$ is the free Lie algebra, $\text{Ass}_n = \mathcal{U}(\text{Lie}(a_n))$ is the free associative algebra “of words”, $\text{tr} : \text{Ass}_n \to \text{Ass}_n / \langle x_1 x_2 \cdots x_m x_1 x_2 \cdots x_m \rangle$ is the “trace” into “cyclic words”, $\text{tr}_n = \text{tr}(\text{Lie}(a_n))$ are all the derivations, and
\[
\text{Lie}(a_n) = \{ D \in \text{Der} : \forall i \exists j \text{ s.t. } D(x_i) = [x_i, a_j] \}
\]
\[
\text{Lie}(a_n) / \langle x_1 x_2 \cdots x_m x_1 x_2 \cdots x_m \rangle
\]
are “tangential derivations”, so $D \leftrightarrow (a_1, \ldots, a_n)$ is a vector space isomorphism $\text{Lie}_n \cong \text{Ass}_n / \langle x_1 x_2 \cdots x_m x_1 x_2 \cdots x_m \rangle$. Finally, $\text{Lie}_n \to \text{tr}_n$ is $\langle a_1, \ldots, a_n \rangle \to \sum_k \text{tr}_k(x_k)$, where for $a \in \text{Ass}_n$, $\partial a \in \text{Ass}_n$ is determined by $a = \sum_k (\partial a) x_k$, and $j : \text{TAut}_n \to \text{tr}_n$ is $j(D) = e^{-1}_D - 1 \cdot D$.

Theorem. Everything matches. (trees) is $\text{Lie}_n \oplus \text{tr}_n$ as Lie algebras, (wheels) is $\text{tr}_n$, (trees) is $\text{tr}_n$, (trees) / $\text{tr}_n$-modules, $D = \exp^{-1} \text{tr}_n$-modules, $D$ is $\exp^{-1}(D)$, and $e^{\text{ad} n - 1 \cdot D} = e^{n D}$.

Differential Operators. Interpret $\mathcal{U}(\text{Lie}(g))$ as tangential differential operators on $\text{Fun}(g)$:
\[
\text{Lie} \otimes \mathcal{U}(\text{Lie}(g))
\]
\[
\exp^{-1}(\text{tr}(\mathcal{U}(\text{Lie}(g))))
\]

Knot-Theoretic statement. There exists a homomorphic expansion $Z$ for trivalent w-tangles. In particular, $Z$ should respect $\text{RA}$ and intertwine annulus and disk unzip:
\[
\text{RA} \otimes \mathcal{U}(\text{Lie}(g))
\]

Diagrammatic statement. Let $R = \exp^{-1} \in \text{Ass}(\uparrow_1)$. There exist $R \in \text{Ass}(\uparrow_1)$ and $V \in \text{Ass}(\uparrow_1)$ so that
\[
R \cdot \mathcal{U}(\text{Lie}(g)) \otimes \mathcal{U}(\text{Lie}(g)) = 1
\]

Algebraic statement. With $I_g : G \to \mathcal{U}(\text{Lie}(g))$, $\mathcal{U}(\text{Lie}(g)) = \mathcal{S}(g^*)$ the obvious projection, with $S$ the antipode of $\mathcal{U}(\text{Lie}(g))$, $W$ the automorphism of $\mathcal{U}(\text{Lie}(g))$ induced by flipping the sign of $g^*$, with $g \in g \otimes g$ the identity element and with $V = e^x \mathcal{U}(\text{Lie}(g)) \mathcal{U}(\text{Lie}(g))$ there exist $\omega \in \mathcal{S}(g^*)$ and $V \in \mathcal{U}(\text{Lie}(g))^{g^2}$ so that
\[
1 \cdot \mathcal{U}(\text{Lie}(g)) \otimes \mathcal{U}(\text{Lie}(g)) = 1
\]
\[
V \cdot \mathcal{U}(\text{Lie}(g)) \otimes \mathcal{U}(\text{Lie}(g)) = 1
\]

Unitary statement. There exists $\omega \in \text{Fun}(\mathcal{U}(g)^G)$ and an (infinite order) tangential differential operator $\text{V}$ defined on $\text{Fun}(\mathcal{U}(g)^G)$ so that
\[
1 \cdot \mathcal{U}(\text{Lie}(g)) \otimes \mathcal{U}(\text{Lie}(g)) = 1
\]
\[
\mathcal{U}(\text{Lie}(g)) \otimes \mathcal{U}(\text{Lie}(g)) = 1
\]

Convolutions statement (Kashiwara-Vergne). Convolutions of invariant functions on a Lie group agree with convolutions of invariant functions on its Lie algebra. More accurately, let $G$ be a finite dimensional Lie group and let $g$ be its Lie algebra, let $f : \mathbb{R} \to \mathcal{U}(\text{Lie}(g))$ be the Jacobian of the exponential map $g \to \mathcal{U}(\text{Lie}(g))$. Then if $f, g \in \text{Fun}(G)$ are Ad-invariant and supported near the identity, then
\[
\Phi(f) \cdot \Phi(g) = \Phi(f \cdot g)
\]

The Orbit Method. By Fourier analysis, the characters of $\text{Fun}(G)^G, *$ correspond to coset/adjoint orbits in $G^*$. By averaging representation matrices and using their lemma to replace intertwining by scalars, to every irreducible representation of $G$ we can assign a character of $\text{Fun}(G)^G, *$. 

http://www.math.toronto.edu/~droron/Talks/Aarhus-1305/ 
http://www.math.toronto.edu/~droron/papers/K3H/ 
http://corbon.net/?title=WKO (joint with Z. Dancso)
Using moves, KTG is generated by ribbon twists and the tetrahedron \( \Delta \):

\[
\begin{array}{c}
\text{plant} \\
\text{agents} \\
\text{connect} \\
\text{isotopy} \\
\text{isotopies} \\
\text{unzip} \\
\text{unzips} \\
\text{isotopy} \\
\text{forget} \\
\end{array}
\]

So \( \Phi \) where \( \Phi \) means \( \Phi \).

Punctures expand to the nearest \( Y \)-vertex:

modulo the relation(s):

\[
\begin{array}{c}
\text{Modulo the relation(s):} \\
\end{array}
\]

Claim. With \( \Phi := Z(\Delta) \), the above relation becomes equivalent to the Drinfel’d’s pentagon of the theory of quasi-Hopf algebras.

Proof. 

\[
\begin{array}{c}
\text{Proof.} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\( \Phi \in \mathcal{A}(\uparrow_3) \} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\( \Phi \in \mathcal{A}(\uparrow_4) \} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\( \Phi \in \mathcal{A}(\uparrow_4) \} \\
\end{array}
\]

\[
\begin{array}{c}
\text{\( \Phi \in \mathcal{A}(\uparrow_4) \} \\
\end{array}
\]

Theorem. The generators of \( \mathcal{K}^w \) can be written in terms of the generators of \( \mathcal{K}^u \) (i.e., given \( \Phi \), can write a formula for \( V \).

Sketch.

\[
\begin{array}{c}
\text{Sketch.} \\
\end{array}
\]

so enough to write any \( T \). Here go:

\[
\begin{array}{c}
\text{Theorem. The Alekseev-Torossian statement is equivalent to the knot-theoretic statement.} \\
\end{array}
\]

Proof. Write \( V = e^{cD}e^{bD}n \) with \( c \in \mathcal{K}_2 \), \( D \in \mathcal{K}_2 \), and \( \omega = e^b \) with \( b \in \mathcal{K}_1 \). Then (1) \( e^{cD}e^{bD} = e^{cD+bD} \), (2) \( e^{cD}e^{bD} = e^{cD+bD} \), and (3) \( e^{cD}e^{bD} = e^{cD+bD} \).