From Stonehenge to Witten Skipping all the Details
Oporto Meeting on Geometry, Topology and Physics, July 2004
Dror Bar–Natan, University of Toronto

It is well known that when the Sun rises on midsummer’s morning over the "Heel Stone" at Stonehenge, its first rays shine right through the open arms of the horseshoe arrangement. Thus astrological lineups, one of the pillars of modern thought, are much older than the famed Gaussian linking number of two knots.

Recall that the latter is itself an astrological construct: one of the standard ways to compute the Gaussian linking number is to place the two knots in space and then count (with signs) the number of shade points cast on one of the knots by the other knot, with the only lighting coming from some fixed distant star.

Thus we consider the generating function of all stellar coincidences:

$$Z(K) := \lim_{N \to \infty} \sum_{\beta=0}^{ \infty} \frac{1}{2^\beta \beta!} (D,K)_\beta D \cdot \text{count with signs}$$

$$\langle D,K \rangle_\beta \text{ pairing of } D \text{ and } K$$

$$D= \begin{matrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{matrix}$$

$$K= \begin{matrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{matrix}$$

$$\Rightarrow \text{ count with signs}$$

$$\text{The Gaussian linking number}$$

$$lk( \begin{matrix} \circ & \circ & \circ \\ \circ & \circ & \circ \\ \circ & \circ & \circ \end{matrix} ) = \frac{1}{2} \sum \text{ vertical chopsticks}$$

$$lk=2$$

$$\text{Carl Friedrich Gauss}$$

$$\text{Dylan Thurston}$$

**Theorem.** Modulo Relations, $Z(K)$ is a knot invariant!

When deforming, catastrophes occur when:

- A plane moves over an intersection point – Solution: Impose IHX,
- An intersection line cuts through the knot – Solution: Impose STU,
- The Gauss curve slides over a star – Solution: Multiply by a framing-dependent counter-term.

It all is perturbative Chern–Simons–Witten theory:

$$\int DA \exp \left[ \frac{i}{4\pi} \int \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right]$$

$$\Rightarrow \sum \text{ Feynman diagram} \quad W_3(D) \sum \text{D diagrams} \quad \sum \text{D diagrams} \sum \text{Feynman diagram}$$

"God created the knots, all else in topology is the work of man."

Richard Feynman

Leopold Kronecker

(modified)

This handout is at http://www.math.toronto.edu/~drorbn/Talks/Oporto–0407

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Knotted Trivalent Graphs, Tetrahedra and Associators

HUJI Topology and Geometry Seminar, November 16, 2000

Dror Bar–Natan

Goal: \( Z: \{ \text{knots} \} \rightarrow \{ \text{chord diagrams} \} / 4T \) so that

\[
\begin{align*}
\begin{array}{c}
\text{The Miller Institute knot}
\end{array}
\end{align*}
\]

Modulo the relation(s):

\[
\begin{array}{c}
\begin{array}{c}
\text{Claim. With } \Phi := Z(\Delta), \text{ the above relation becomes equivalent to the Drinfel’d’s pentagon of the theory of quasi Hopf algebras.}
\end{array}
\end{array}
\]

Proof.

\[
\begin{array}{c}
\begin{array}{c}
\text{Further directions:}
\end{array}
\end{array}
\]

1. Relations with perturbative Chern–Simons theory.
2. Relations with the theory of \( 6j \) symbols
3. Relations with the Turaev–Viro invariants.
4. Can this be used to prove the Witten asymptotics conjecture?
5. Does this extend/improve Drinfel’d’s theory of associators?

This handout is at http://www.ma.huji.ac.il/~drorbn/Talks/HUJI–001116