0. Plan: Today: preliminaries & aside, the cheese grater principle.

Next time: The actual proof.

1. Restate the theorem and the "detached wheels/hairy y" claim.

2. The Euler trick

An Euler Interlude. If you know brackets, how do you test exponentials? When’s $e^A e^B = e^{C} e^{D}$?

Bad Idea. Take log and use BCH. You’ll want to cry.

Clever Idea. Let $E$ be the Euler derivation, which multiplies each element by its degree (e.g. on $\mathbb{Q}[\phi]$, $E\phi = \phi \partial \phi$, so $E e^{\phi} = \phi e^{\phi}$). Apply $E : = \zeta^{-1} E \zeta$: $E (e^{A} e^{B}) = e^{-B} e^{-A} (e^{A} e^{B} e^{A} e^{B}) = e^{A} e^{B} e^{A} e^{B} + B = e^{-AD}(A) + B$.

b. The BCH formula

Thm. In $FA(x,y)$, $e^{x+y} = e^\phi = e^{bch(x,y)}$,

where $\phi$ is a Lie series.

* Good for making Lie groups out of Lie algebras

* will be extremely important for us later on.

$\phi = \log e^{x+y}$, but why is it a Lie series?

Proof. Apply $E$, get

$$E e^{\phi} = e^{-AD}(x) + y \in \text{Lie}(x,y)$$

Aside: how compute $E e^{\phi}$? In general, $De^{\phi}$?

Aside 2: What’s $\text{det}^\phi = \text{det}_e$, where $\text{exp}: \mathfrak{m} \to \mathfrak{m}$?
\[ e^{A + eB} = e^A + e \cdot \frac{e}{6} + o(e^2) \]

\[ z = \sum_{n=1}^{\infty} A A \cdots B \cdots A \quad BA = AB - adA (8) \]

\[ = e^A \cdot F_{e^A} (e^A) = \frac{e^{-adA}}{adA} (B) \quad \frac{e^{-adA} - 1}{-adA} \]

So \[ 1 - e^{-ad\phi} \cdot (E\phi) = e^{-x+y} \in \text{Lie}(x,y) \]

\[ E\phi = \frac{1}{2} [\phi, E\phi] + \frac{1}{6} [\phi, [\phi, E\phi]] - \cdots = e^{-x+y} \]

Claim \( \phi \) exists and is unique and belongs to \( \text{Lie}(x,y) \).

Proof: In the spirit of the inverse function thm, write \( \phi_x = \phi + y \ldots \)

---

3. so apply \( \tilde{E} \) to \( z = \)

\[ z^{-1} \quad \text{and yet} \quad z \quad z^{-1} \]
\[ e^{ad-b} = e^a e^{-b} \]

\[ e^x = \frac{e^x - 1}{x} \times x \]

\[ e^b a = e^{ad-b(a)} e^b = ab + e^{ad-b(a)} \cdot e^b \]

4. The cheese-grator approach.