Let $n$ be an integer. How many ways are there to tile the $n$-staircase with exactly $n$ rectangles?

$$y'' + p(x)y' + q(x)y = 0 \quad \frac{v_1 = y}{v_2 = y'} \quad V' = \begin{pmatrix} 0 & 1 \\ -q & p \end{pmatrix} V$$

Let $y_1, y_2$ be independent solutions, $W = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$

$$W' = -PW \quad \text{(*Abel's Thm*)}$$

Example. $y'' + y = 0$

**Problem:** Find a power series $y(x) = \sum_{k=0}^{\infty} a_k(x-x_0)^k$ that solves the equation $y' = f(x, y)$

**Motivation:**
1. Minor: Combinatorics.
2. Major: Power series are easy at large scale, but excellent at small scales. See QED $w/ x = \frac{1}{\sqrt{x}}$.

```
In[1]: PowerSeriesSolve[f_, xo_, y0_, n_] := 
    Do[
      phi[k] = y0 + Integrate[Normal[Series[f /. y -> phi[k-1], {x, xo, k-1}]] /. x -> t, 
        {k, 1, n}];
      phi[0] = y0;
    ];

In[2]: PowerSeriesSolve[Sqrt[1 + x^2], 0, 0, 5]
Out[2]= x - \frac{x^3}{6} + \frac{x^5}{120}

In[3]: PowerSeriesSolve2[f_, xo_, y0_, n_] := Module[{phi = y0},
    Do[
      phi[k] = D[f /. y -> phi[k-1], {x, k-1}] /. x -> xo,
        {k, 1, n}];
    phi[0] = y0;
    ];

In[4]: PowerSeriesSolve2[Sqrt[1 + x^2], 0, 0, 10]
Out[4]= x + \frac{x^3}{6} + \frac{x^5}{50} + \frac{x^7}{5040} + \frac{x^9}{362880}

In[5]: Series[Sinh[x], {x, 0, 10}]
Out[5]= x + \frac{x^3}{6} + \frac{x^5}{50} + \frac{x^7}{5040} + O[x^{11}]
```

**Little on power series.**
Little on power series.

**Theorem 1.** Given \( \sum a_n x^n \), the “radius of convergence,” \( R \), is

- **absolutely convergent** if \( |x| < R \)
- diverges if \( |x| > R \).

If \( |x| = R \), “it depends.”

\[ R = \sup \{ r : a_n r^n \to 0 \} \]

- \( = \sup \{ r : |a_n r^n| \text{ is bounded} \} \)

**Theorem 2.** (Loose) 1. If \( f \) has a formula, it has a natural extension to \( C \).

2. In that case, \( R \) is the distance from \( 0 \) to the nearest point in \( C \) in which the formula fails.

**Examples.** 1. \( \sin x = \frac{e^{ix} - e^{-ix}}{2i} \)

2. \( \frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 \ldots \)

**Problem.** Given \( y'' + p(x)y' + q(x)y = g(x) \), find a power series \( y = \sum a_n x^n \) that solves this eqn.

Do the \( y'' + y = 0 \) example.

Do the Airy example \( y'' = xy \)

State Fuchs’ Theorem: The series for \( y(x) \) converges at least the last \( R \) for \( p(x) \).

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**Mathematica Code:**

```mathematica
In[1]:= NDSolve[y''[x] - x y[x] == 0, y[x], {x, -10, 3}];
In[2]:= NDSolve[y''[x] - x y[x] == 1, y[x], {x, -10, 3}];
In[3]:= Plot[Evaluate[y[x] /. In[1], {x, -10, 3}]]
```

---

**Plot:**

![Graph of solutions](image-url)