Riddle: How many binary trees with n leaves are there?

Claim: If $\psi(t) = A(t)\psi(t)$, then either $\psi$ is regular for all $t$ or singular for all $t$.

**PE1:** Use existence & uniqueness

Debts: 1. Make extra interval length explicit.
2. PE2 using the Wronskian $W$.

Claim: Solutions to $V'(t) = A(t)V(t)$ exist and are unique wherever $A(t)$ is continuous.

Proof: Lipschitz is non-issue. Usual existence/uniqueness limited to $\frac{1}{M}$ if $|A_{ij}| < M$ on $I = [-4000, 4000]$, then $M < N\lambda b$ on $I$ so $\frac{1}{M} > \frac{1}{\lambda N} = \varepsilon$ so wherever a solution exists, it exists an $\varepsilon$ further.

**PE2:** Use the Wronskian $W = \det \psi(t)$:

$W(t+\varepsilon) = \det(\psi(t+\varepsilon)) : \det(\psi(t) + \varepsilon \psi') = \det(\psi + \varepsilon A \psi)$

$= \det(I + \varepsilon A) \det \psi = (1 + \varepsilon (tA)) W$

So $W' = (tA) W$ so $W = \exp(tA) W(t_0)$

Power Series, an unusual motivation.

1. Power series are keepers of combinatorial information.
2. Recursion relations $\leftrightarrow$ differential eqns.

$A_n = \frac{1}{n+1} \binom{2n}{n} \quad \Rightarrow \quad F(x) = \sum A_n x^n$

$C_n = \binom{\# of below - diagonal paths}{\# from \(l, p\) to \(m, q\)} \quad \Rightarrow \quad G(x) = \sum C_n x^n$

"Catalan numbers" is final: commute 2nd 1st...
"Catalan numbers"

\[ C_0 = 1 \]

\[ C_{n+1} = \sum_{k=0}^{n} C_k \cdot C_{n-k} \]

\[ xG^2 - G + 1 = 0 \]
\[ G = \frac{1 - \sqrt{1 - 4x}}{2x} \]

New analyze \( F \):
\[ A_n = \frac{(2n)!}{n!(n+1)!} \]

\[ A_{n+1} = \frac{(2n+2)(2n+1)}{(n+1)(n+2)} A_n = 2 \frac{2n+1}{n+2} A_n \]

\[ \sum_{n=0}^{\infty} (n+1) A_n x^n - 1 = \sum_{n=0}^{\infty} (n+2) A_{n+1} x^{n+1} = x \sum_{n=0}^{\infty} (n+2) A_n x^n \]

\[ xF' + F - 1 = x \left( 4xF' + 2F \right) \]

\[ F(0) = 1 \quad 0 = (4x^2 - x)F' + (2x-1)F + 1 \]

Thus \( F = G \) &
\[ C_n = \frac{1}{n+1} \binom{2n}{n} \]

\[ r = \frac{1 - \sqrt{1 - 4x}}{2x} \]
\[ \frac{1 - \sqrt{1 - 4x}}{2x} \]

\text{series}[r, \{x, 0, 10\}]

\[ 1 - x + 2x^2 - 5x^3 + 14x^4 - 42x^5 - 132x^6 + 429x^7 - 1430x^8 + 4862x^9 - 16796x^{10} + O[x]^{11} \]

\[ x(4x - 1)D[f, x] + 2xf - f + 1 // \text{Simplify} \]