Compositions and Multiplications

\[ V/F, \text{ basis } \beta=(v_1,...,v_n) \quad W/F, \text{ basis } \gamma=(w_1,...,w_m) \]

Let \( T : V \to W \) be a linear transformation. Given a basis \( \beta \) for \( V \) and a basis \( \gamma \) for \( W \), we can construct a matrix representation of \( T \) with respect to these bases.

\[ A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \]

\[ T \mapsto [T]_\beta^\gamma = A \]

If you know \( A \) and \( B \) can you and \( C \) derive \( C \)?

**Definition** \( A \in M_{mn}, \ B \in M_{np} \) \( AB \in M_{mp} \) by \( (AB)_{ik} = \sum_{j=1}^n A_{ij} B_{jk} \)

**Example**

\[
\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} = \ldots
\]

I should have had another example. \( TA \ V = AV \) where \( V \) is regarded as an \( n \times 1 \) matrix.

**Theorem** \( [T \circ S]^\gamma_\alpha = [T]^\gamma_\beta [S]^\beta_\alpha \)

**Example** \( T_\beta \circ T_\alpha = T_{\beta + \alpha} \) for rotations.

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**The good and the bad about “matrix algebra”**

<table>
<thead>
<tr>
<th>Good</th>
<th>Bad</th>
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<tbody>
<tr>
<td>1. ( A+B = B+A ) ( (A+B)+C = A+(B+C) ) (Basically all works for addition)</td>
<td>1. Addition is defined only for matrices of same dimensions</td>
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<tr>
<td>2. ( A \circ (B \circ C) = (A \circ B) \circ C )</td>
<td>2. Multiplication is defined only if “right” dimensional matches, &amp; produces an output of yet other dimensions</td>
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<td>3. If ( A \circ I = I ), then ( A \circ I = I )</td>
<td>3. ( A^{-1} ) may not exist even if ( A \neq 0 )</td>
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<td>4. ( (A \circ B) \circ C = AC + BC )</td>
<td>4. Generally, ( AB \neq BA ), even when both make sense.</td>
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| \( A(B + C) = AB + AC \) |\]

**Proposition** Given \( V \xrightarrow{\phi} V \rightarrow W \xrightarrow{\psi} W' \) with invertible \( \psi \).

1. \( \text{rank } T = \text{rank } P_{\phi}TQ \) [enough that \( Q \) surjective \& \( P \) injective]
2. \( \psi \mid_{\text{rank } T} \circ \text{im}(T) = \text{im}(T') = C \) basis \( \{w_i = T(v_i)\}_{i=1}^r \)
3. \( \psi \mid_{\text{rank } T} \circ \text{im}(T) = \text{im}(T') = C' \) basis \( \{w'_i = P(w_i)\}_{i=1}^r \)
4. \( \text{Nad: } 1. wi' \in \text{im } T' \); meaning \( \exists v' \in V' \) s.t. \( w'_i = T'v'_i \)
5. \( \text{Nad: } 2. w'_i \text{ span } C' \)
6. \( \text{Nad: } 3. w'_i \text{ are lin. indep.} \)

**Def:** If \( A \in \mathbb{F}^{m \times n} \), let \( \text{rank } A = \text{rank } TA \), where

- \( T_a \text{ is the standard } T_a : \mathbb{F}^n \rightarrow \mathbb{F}^m \)

**Comment 1:** \( \text{rank } [T]_p = \text{rank } T \)

**Comment 2:** \( \text{rank } A = \text{rank } P_{\phi}Q \) whenever

- \( P \in \mathbb{F}^{m \times m} \) \& \( Q \in \mathbb{F}^{n \times n} \) are invertible.

- Look for \( P \times Q \) that will make

  - \( P_{\phi}Q \) “simple” than \( A \).

**Claim:** \( \text{rank } \left( \begin{bmatrix} I_{m-k} & 0 \\ 0 & 0 \end{bmatrix} \right) = k \)

Examples of good \( P \times Q \): 1. Interchanging rows/columns.
2. Multiplying r/c by a scalar.
3. Adding a multiple of one r/c to another.