Riddle Along. (Thanks, ….) Five piles of
100 gram gold coins are given, but it is known
that the coins in one of the piles are fakes,
and weigh only 99 grams. Find which pile it
is, with only one use of an accurate
scale.

Discussion: next class.

Lemma (the replacement lemma)

<table>
<thead>
<tr>
<th>G</th>
<th>V, span(G) = V, L lin indep</th>
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<tbody>
<tr>
<td>\Rightarrow</td>
<td>\forall \lambda \in \mathbb{R} \ V \ \lambda L \</td>
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Corollaries: 1. If V has a finite basis B, then every
other basis B2 of V is also finite & |B| = |B|.

2. “dim V” makes sense.

3. Assume dim V = n. Then

A. If G generates V, G \subset G is an element of G is a basis.
   if also G \subset G, then G itself is a basis.

B. If L is linearly indep in V, \exists L \subset L, |
   if also L \subset L, \ L \ is a basis.
   if also L \subset L, \ L \ can be extended to a

4. If V is finite-dimensional and \subseteq V is a
   subset A, then W is f.d. and \text{dim} W \leq \text{dim} V.
   If also \text{dim} W = \text{dim} V, then \text{dim} W = \text{dim} V.
   \text{If also} \ \text{dim} W < \text{dim} V, \text{then any basis of} \ W \text{ not done}
can be extended to a basis of \( V \).

The Lagrange interpolation formula:

Let \( x_i \) be distinct pts in \( \mathbb{R}/F \) for \( i = 1, \ldots, n+1 \).
Let \( y_i \) be any pts in \( \mathbb{R}/F \).

Q: Can you find a polynomial \( p \in \mathcal{P}_n(\mathbb{R}) \) s.t. \( p(x_i) = y_i \)?

**Solution**

\[
p_i(x) = \prod_{j \neq i} (x - x_j)
\]

Then \( p_i(x_j) = \delta_{ij} \)

Set \( p(x) = \sum_{i=0}^{n} \frac{y_i p_i(x)}{p'_i(x_j)} \)

Then \( p(x) = \sum_{i=0}^{n} y_i p_i(x) \) satisfies \( p(x_i) = y_i \)

\( \beta = \{ p_0, \ldots, p_{n+1} \} \) is lin. indep.

\( \Rightarrow \beta \) is a basis

* Every \( f \in \mathcal{P}_n(\mathbb{R}) \) can be expressed as a lin. comb. of the \( p_i \) in a unique way.

* If \( q(x) \) also satisfies \( q(x_i) = y_i \), then \( q(x) = p(x) \).

* Therefore, the solution to our problem is unique.

* Aside: If \( \forall x \in \mathbb{R} \), \( p(x) = 0 \), then \( \gamma = 0 \)

(since a non-zero polynomial of degree \( n \) has at most \( n \) roots.)