**Riddle Along.**

**Read Along.** 1. Y - 1.6.

**Web Fact:** (not visible) = (doesn’t exist).

**Life Fact:** No “teaching over email.”

**Reminders:** We seek “basis” i.e. $\text{j} \text{ c. j span j “generates”}

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**Def:** A subset $S \subset V$ is “lin dep” if it is “wasteful.”

I.e., if $\exists \alpha \in F$ not all 0 but $S$ sits: $\sum \alpha \text{v} = 0$.

Otherwise, it is “lin. indy.”

**Examples:** \[
\{v_1, v_2\}, \{w_1, (1), (2), (3)\}
\]

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**Conjects 1.** $\emptyset$ is lin. indy.

2. $\{v\} \text{ is lin indy iff } v \neq 0$.

3. Suppose $S_1, S_2 \subset V$. Then
   - If $S_1$ is dep, so is $S_2$
   - If $S_2$ is indy, so is $S_1$

4. If $S$ is lin indy in $V$ and $v \in V \setminus S$, then $S \cup \{v\}$ is lin. dep. iff $v \in \text{span}(S)$.

**Def:** Basis $B \subset V$.

**Examples:** 1. $\emptyset$ for $\{0\}$.

2. $e_i$ for $E^n$.

3. $e^{ij}$ for $M_{\text{sym}}(E)$.

4. $(1, x, \ldots, x^n)$ for $P_n(E)$.

5. $(1, x, \ldots)$ for $P(E)$.

6. $\{(1), (-1)\}$ for $\mathbb{R}^2$.

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**Thm:** A subset $\beta \subset V$ is a basis iff every $v \in V$ can be expressed in a unique way as a l.c. of elements of $\beta$.

**Done**
There is a subset $B \subseteq V$ which is a basis of $V$.

**Our first non-language theorem:**

**Thm** If a v.s. $V$ has a finite basis, then every other basis of $V$ has the same number of elements in it.

**Def** If $V$ has a finite basis, we say that it is "finite-dimensional" and let $\dim V := \left(\text{the number of elements in (any) basis of } V\right)$.

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**Lemma** (the replacement lemma)

Let $G \subseteq V$ be a linearly independent subset of $V$. Then, $G$ is a basis for $V$.

**Proof of Theorem from Lemma.**

Informal proof of Lemma. First of all, if $\sum a_i v_i = 0$, the any vector that appears in this dependency with non-zero coeff is a l.c. of the others.

Now, let $G$ be a basis of $V$.

\[ \dim V = \text{the number of elements in (any) basis of } V \]

Formal proof: Induction on $\dim L$. $\dim L = 0$: trivial.

Now, $\dim L = m + 1$.

Let $L = \{v_1, \ldots, v_{m+1}\}$. Use $L' = \{v_1, \ldots, v_m\}$.

Find $H = \{u_1, \ldots, u_{m-1}\} \subseteq G$ s.t. $u_i \in \text{span}(v_1, \ldots, v_m)$ and $L'' = \{u_1, \ldots, u_{m-1}, v_{m+1}\}$ spans $V$.

Write $v_{m+1} = a_{-1} u_1 + \ldots + a_{-m} u_{m-1} + b_1 v_1 + \ldots + b_m v_m$.

\[ \therefore \text{Not all } a_i = 0, \text{ so } n > m, \text{ so } m + 1 \in \mathbb{N} \]
... w.l.o.g. $a_i \neq 0$, so $u_i \in \text{span}(u_2, \ldots, u_{n-m}, v_i, \ldots, v_{n+1})$.

so take $H = \{u_2, \ldots, u_{n-m}\}$.

Corollaries: 1. If $V$ has a finite basis $\beta$, then any other basis $\beta_2$ of $V$ is also finite and $|\beta_1| = |\beta_2|$.

2. “dim $V$” makes sense.

3. Assume dim $V = n$. Then

a. If $G$ generates $V$, $|G| = n$ if also $|G| = n$,

then $G$ is a basis.

b. If $L$ is linearly independent in $V$, then $|L| < n$.

if also $|L| = n$, $L$ is a basis.

if also $|L| < n$, $L$ can be extended to a basis.

4. If $V$ is finite-dimensional and $W \subset V$ is a subspace, then $W$ is finite-dimensional, $\dim W \leq \dim V$.

If also $\dim W = \dim V$, then $W = V$.

If also $\dim W < \dim V$, then any basis of $W$ can be extended to a basis of $V$. 