UNIVERSITY OF TORONTO

Faculty of Arts and Sciences

DECEMBER EXAMINATIONS 2012

Math 240H1 Algebra I — Final Exam

Dror Bar-Natan

December 13, 2012

Solve all of the following 5 questions. The questions carry equal weight though different parts of the same question may be weighted differently.

Duration. You have 3 hours to write this exam.

Allowed Material. Basic calculators, not capable of displaying text or sounding speech.

Good Luck!

Problem 1. It is given that W_1 and W_2 are subspaces of the same vector spaces V. Prove that their union $W = W_1 \cup W_2$ is also a subspace of V if and only if $W_1 \subset W_2$ or $W_2 \subset W_1$.

Tip. "If and only if" means that there are two things to prove.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Problem 2. Prove the "replacement lemma": Let G be a set of g vectors that spans some vector space V and let L be some set of l linearly independent vectors in V (where g and l are both finite). Then $g \geq l$ and there is a subset R of G, consisting of $r := g - l$ vectors, so that $\operatorname{span}(R \cup L) = V$.

Tip. As always in math exams, when proving a theorem you may freely assume anything that preceded it but you may not assume anything that followed it.

Tip. Neatness, cleanliness and organization count, here and everywhere else!

Problem 3. Recall that the real numbers $\mathbb{R}$ are a vector space over the field $\mathbb{Q}$ of rational numbers. Let V be the subspace of $\mathbb{R}$ given by $V = \{a + b\sqrt{3} \colon a, b \in \mathbb{Q}\}$.

1. Find a basis β for V over $\mathbb{Q}$.
2. Let $T \colon V \to V$ be the linear operator defined by $Tx = \sqrt{3}x$. Find the matrix representing T relative to the basis β you found in the previous part of this question.

Tip (added after exam). Note that β is the basis of both the domain and the target space of T.

Problem 4. Let M be the 5×5 "multiplication table" matrix shown below, let A be the 5×5 "addition table" matrix shown below, and let S be the 6×6 "snakes and ladders" matrix shown below:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 4 & 6 & 8 \\ 0 & 3 & 6 & 9 & 12 \\ 0 & 4 & 8 & 12 & 16 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{pmatrix} \qquad \begin{pmatrix} 36 & 35 & 34 & 33 & 32 & 31 \\ 25 & 26 & 27 & 28 & 29 & 30 \\ 24 & 23 & 22 & 21 & 20 & 19 \\ 13 & 14 & 15 & 16 & 17 & 18 \\ 12 & 11 & 10 & 9 & 8 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$M \qquad\qquad\qquad A \qquad\qquad\qquad S$$

1. Bring the matrices M and A to reduced row echelon form.
2. Determine the ranks of M and of A.
3. Show that every row of the matrix S is a linear combination of its bottom row and the row $(1\ 1\ 1\ 1\ 1\ 1)$.
4. Deduce that the rank of S is at most 2.
5. Show that the rank of S cannot be 0 or 1, and hence it must be 2.

Note (added after exam). People preferred computation and didn't quite get the point of determining the rank of S by the 3–5 sequence resulting in a harder-to-grade question.

Problem 5. Let A be the matrix $A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$.

1. Compute $\det(A - \lambda I)$.
2. Find the eigenvalues λ_1 and λ_2 of A.
3. Find their corresponding eigenvectors v_1 and v_2.
4. Find a matrix C for which $AC = CD$, where $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.
5. Compute the inverse of C.
6. Compute A^7 by computing CD^7C^{-1}.

Note (added after exam). Instead of A^7, I should have asked for a general formula for A^n, for $n \in \mathbb{N}$.

Good Luck!