October 27, hour 21: Rings, ideals, isomorphism theorems, prime
and maximal ideals

Goal. 1. Rings, ideals, isomorphisms.

2. Prime & maximal ideals, domains and fields.

Definition 2.1.1. A ring consists of a set $R$ together with binary operations $+$ and $\cdot$ satisfying:

1. $(R, +)$ forms an abelian group,
2. $(a \cdot b) \cdot c = a \cdot (b \cdot c) \forall a, b, c \in R,$
3. $\exists 1 \neq 0 \in R$ such that $a \cdot 1 = 1 \cdot a = a \forall a \in R,$ and
4. $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c \forall a, b, c \in R.$

Also define:

Commutative ring.

Examples. $\mathbb{Z}, R[x], M_{nxn}(R)$

Morphisms:

Examples:
1. $\mathbb{Z} \rightarrow \mathbb{Z}/n$
2. $R \rightarrow R[x]$ as degree
3. $R \rightarrow M_{nxn}(R)$ as $\text{deg}$
4. $\forall_{u \in R}[x] \rightarrow R$ (if $R$ is commutative)
5. $M_{nxn}(R[x]) \cong M_{nxn}(R)[x]

Added Dec 2012: Perhaps I should have proven Cayley-Hamilton right here:

$det(\lambda I - A) = \det(\lambda I - A)(\lambda I - A) = (\lambda B_i \cdot t^i)(\lambda I - A),$

now substitute $t = A.$ The $B_i$'s commute with $A$
because $(\lambda I - A) \det(\lambda I - A) = \det(\lambda I - A)(\lambda I - A).$

im, subring, ker, ideal.

Q. Is every ideal a quotient?

Ans. Define $R/I.$

The Isomorphism theorems.

1. $\phi: R \rightarrow S \Rightarrow R/\ker(\phi) = \text{im} \phi.$

2. $A + I = A / A \cdot I \quad A \in R \text{ subring}, I \in R \text{ ideal}.
3. \( I \subseteq J \subseteq R \) ideals \( \Rightarrow \frac{R/I}{J/I} \cong R/J \)

4. Given an ideal \( I \) of \( R \), there is a bijection between ideals \( I \subseteq J \subseteq R \) & ideals of \( R/I \).