October 13, hour 15: Semi-direct products

October-11-11
5:51 PM

Agenda. ! Semi-direct products & examples. / web Comments:

1. Filenames must begin w/ 11-1100
2. what's not linked doesn't exist.

Read Along. Selick 1.8, 1.10.

Riddle Along.

1. Can you find uncountably many nearly-disjoint $[\forall \alpha, \beta\ |A_\alpha \cap A_\beta| < \infty]$ subsets of $\mathbb{N}$?
2. Can you find an uncountable chain $[\forall \alpha, \beta, (A_\alpha \subset A_\beta) \vee (A_\beta \subset A_\alpha)]$ of subsets of $\mathbb{N}$?

Semi-Direct Products. Given N, H & $\phi: H \xrightarrow{\text{mor}} \operatorname{Aut}(N)$,

$$N \rtimes_\phi H := \left(N \times H,\ (n_1, h_1)\cdot(n_2 h_2) = (n_1 \phi_{h_1}(n_2), h_1 h_2)\right)$$

Thm. 1. $G := N \rtimes_\phi H$ is a group, $H < G$, $N \triangleleft G$ and $G/N \cong H$, and $G = NH$.

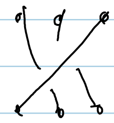
2. If $G = NH$, $N \triangleleft G$, $H < G$, $H \cap N = \emptyset$ then $G \cong N \rtimes_\phi H$.

Small Examples. 1. $D_{2n} = \mathbb{Z}/n \rtimes \{\pm 1\}$

2. $\{ax+b\} = \mathbb{R}^+_b \rtimes \mathbb{R}^\times_a$

3. $\{Ax+b : A \in GL(V), b \in V\} = V_b \rtimes GL(V)_A$

4. "The Poincare Relativity Group" $= \mathbb{R}^4 \rtimes SO(3,1)$

Big Example. $B_n = \pi_1\left((\mathbb{C}^n - \{\text{diags}\})/S_n\right) =$ 

$$B_n = \left\langle \sigma_1, \dots \sigma_{n-1} : \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j|>1 \end{array} \right\rangle$$

an aside on free groups, generators & relations.

Done line

$\pi : B_n \longrightarrow S_n$ $\qquad PB_n = \ker \pi$

$PB_n \triangleleft B_n$ yet not $B = PB_n \rtimes S_n$

 Two reasons why I like this one:

$PB_n \triangleleft B_n$ yet not $B_n = PB_n \rtimes S_n$

Two reasons why I like this one:
1. knotted $20's.
2. Borromean.

$\rho : PB_n \to PB_{n-1}$ $\quad \ker\rho \cong F_{n-1}$ and

$$PB_n = F_{n-1} \rtimes PB_{n-1} = F_{n-1} \rtimes (F_{n-2} \rtimes (\dots (F_2 \rtimes \mathbb{Z}) \dots))$$

Groups of order 21. $\mathbb{Z}/21$, $\mathbb{Z}/7 \rtimes \mathbb{Z}/3 = \langle x \rangle \rtimes \langle y \rangle$

$\mathrm{Aut}(\mathbb{Z}/7) = \mathbb{Z}/6 = \langle \phi_3 \rangle$; $\phi_3(x) = x^3$; $x^y = x$ or x^2 or x^4 (isomorphic)

(iso: if $x^y = x^2$ & $\bar{y} = y^2$ then $x^{\bar{y}} = x^4$)

Groups of order 12. If $|G| = 12$, $P_4 = \mathbb{Z}/4$ or $(\mathbb{Z}/2)^2$, $P_3 = \mathbb{Z}/3$, and at least one of those is normal, for there's not enough room for 4 P_3 & 3 P_4's. So G is a semi-direct product: $\mathbb{Z}/4 \rtimes \mathbb{Z}_3$: must be $\mathbb{Z}/4 \times \mathbb{Z}/3 = \mathbb{Z}/12$

$(\mathbb{Z}/2 \times \mathbb{Z}/2) \rtimes \mathbb{Z}_3$: Either direct; $\mathbb{Z}/2 \times \mathbb{Z}/6$

or the fun action of $\mathbb{Z}/3$ on $(\mathbb{Z}/2)^2$, giving A_4

$\mathbb{Z}/3 = \langle (123) \rangle$; $(\mathbb{Z}/2)^2 = \{e, (12)(34), (13)(24), (14)(23)\}$

$\mathbb{Z}/3 \rtimes (\mathbb{Z}/2 \times \mathbb{Z}/2)$: Either direct or $D_6 \times \mathbb{Z}/2 = D_{12}$

$\mathbb{Z}/3 \rtimes \mathbb{Z}/4$: Either direct or $\mathbb{Z}/3 \rtimes \mathbb{Z}/4$

Added Jan 28, 2013: I should have added a comment / exercise, following Goodwillie's comment in http://mathoverflow.net/questions/119803/relation-between-groups-and-classifying-spaces, that the semi-direct product of a group with itself using the conjugation action is isomorphic to the direct product:

Goodwillies' M.O. comment.

$$\begin{array}{ccc} G^2 \times G^2 & \xrightarrow{m_2} & G^2 \\ \varphi\times\varphi \downarrow & & \downarrow \varphi \\ G^2 \times G^2 & \longrightarrow & G^2 \end{array}$$

$$(g_1,g_2)\times(h_1,h_2) \xmapsto{\varphi^{-1}\times\varphi^{-1}} (g_1,g_1g_2)\times(h_1,h_1h_2)$$

$$\xrightarrow{m_2} (g_1h_1,\ g_1g_2h_1h_2) \xmapsto{\varphi}$$

$$\xrightarrow{\varphi} (g_1h_1,\ h_1^{-1}g_2h_1\cdot h_2) = (g_1h_1,\ g_2^{h_1},\ h_2)$$

$$\boxed{\begin{array}{c}(g_1,g_2)\times(h_1,h_2)\mapsto(g_1h_1,\ g_2^{h_1}h_2) \\ G\ltimes G\end{array}}$$

$$\varphi(g_1,g_2) = (e,g_2)\cdot(g_1,g_1^{-1})$$

$$= (g_1,\ g_1^{-1}g_2)$$

$$\varphi^{-1}(g_1,g_2) = (g_1,\ g_1g_2)$$

$$(g,g^{-1})\times(h,h^{-1}) \mapsto (gh,\ (g^{-1})^h h^{-1})$$

$$= (gh,\ h^{-1}g^{-1})$$