October 13, hour 15: Semi-direct products

Agenda. 1. Semi-direct products & examples. 2. Comments:

Read Along. Selick 1.8, 1.10.

Riddle Along.

1. Can you find uncountably many nearly-disjoint \([A_x \cap A_y | < \infty]\) subsets of \(N\)?

2. Can you find an uncountable chain \([A_x \in (A_x A_y v (A_y A_x)]]\) of subsets of \(N\)?

Semi-Direct Products. Given \(N, H \times \phi : H \rightarrow \text{Aut}(N)\),

\[N \rtimes H := \left\{ (N \times H, (n_1, h_1) \cdot (n_2, h_2) = (n_1 \phi_{n_1}(n_2), h_1 h_2) \right\}\]

Thm. 1. \(G = N \rtimes H\) is a group, \(H < G, \forall G\) and \(G/N \cong H\) and \(G = NH\).

2. If \(G = NH, \forall G, H < G, H \cap N = \emptyset\) then \(G \cong N \rtimes H\).

Small Examples. 1. \(D_{2n} = \mathbb{Z}/n \times \{\pm 1\}\)

2. \([ax + b] = \mathbb{R}^*_b \times \mathbb{R}^*_a\)

3. \([ax + b : \text{AGL}(V)] = V_b \times \text{AGL}(V)_a\)

4. "The Poincaré Relativity Group," \(= \mathbb{R}^4 \times SO(3, 1)\)

Big Example. \(B_n = T_1((C_2 \times \text{Sym}(n))/S_n) = \sqcup_{i \neq j}^{n} \mathbb{Z}/i^{j+1} \times \mathbb{Z}/j^{i+1}\)

\(B_n = \langle 0_1, \ldots, 0_{n-1} : 0_i^{i+1} = 0_i^{i+1}, 0_{i-j}^{i+1} = 0_{j-i}^{i+1} \rangle\)

\(T_T : B_n \rightarrow S_n, PB_n = \mathbb{R}^* T_T\)

\(PB_n \cap B_n \neq \emptyset, \text{ yet } R = PB_n \times S_n\)
\[ PB_n \not\subseteq PB_n \quad \text{yet not} \quad B_n = PB_n \times S_n \]

\[ j: PB_n \to PB_{n-1} \quad \ker j = F_{n-1} \quad \text{and} \]

\[ PB_n = F_{n-1} \times PB_{n-1} = F_{n-1} \times F_{n-2} \times (F_{n-2} \times \ldots \times (F_2 \times \mathbb{Z}).) \]

**Groups of order 21.** \( \mathbb{Z}/21 \), \( \mathbb{Z}/7 \times \mathbb{Z}/3 = \langle x \rangle \times \langle y \rangle \)

\[ \text{Aut} (\mathbb{Z}/21) = \mathbb{Z}/6 = \langle \phi_3 \rangle \quad \phi_3(x) = x^3, \quad y^7 = x \text{ or } y^2 \text{ or } x^4 \]

(asso: if \( x^2 = y^2 \) \& \( y = y^2 \) then \( x^3 = y^4 \))

**Groups of order 12.** If \( 161 = 12 \), \( P_4 = \mathbb{Z}/4 \) or \( (\mathbb{Z}/2)^3 \), \( P_3 = \mathbb{Z}/3 \)

and at least one of these is normal, for there is not enough room for \( 4 \) \( P_3 \) \& 3 \( P_4 \)'s. So \( G \) is a semi-direct product: \( \mathbb{Z}/4 \times \mathbb{Z}/3 \) : must be \( \mathbb{Z}/4 \times \mathbb{Z}/3 = \mathbb{Z}/12 \)

\((\mathbb{Z}/2 \times \mathbb{Z}/2) \times \mathbb{Z}/3\) : Either direct \( \mathbb{Z}/2 \times \mathbb{Z}/6 \)

or the fun action of \( \mathbb{Z}/3 \) on \((\mathbb{Z}/2)^2\), giving \( \langle (123), 0 \rangle \)

\[ \text{or} \quad (12)(34) \quad (13)(24) \quad (14)(13) \]

\( \mathbb{Z}/3 \times (\mathbb{Z}/2 \times \mathbb{Z}/2) \) : Either direct or \( D_6 \times \mathbb{Z}/2 = \mathbb{Z}/2 \)

\( \mathbb{Z}/3 \times \mathbb{Z}/4 \) : Either direct or \( \mathbb{Z}/3 \times \mathbb{Z}/4 \)

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Added Jan 28, 2013: I should have added a comment / exercise, following Goodwillie's comment in [http://mathoverflow.net/questions/119803/relation-between-groups-and-classifying-spaces](http://mathoverflow.net/questions/119803/relation-between-groups-and-classifying-spaces), that the semi-direct product of a group with itself using the conjugation action is isomorphic to the direct product:
A group with itself using the conjugation action is isomorphic to the direct product:

\[ G \times G \]

\[ (g, g) \rightarrow (g, g') \rightarrow (g, g'g) \]

\[ \psi(g, g') = (g, g') \cdot (g, g'^{-1}) \]

\[ \psi^{-1}(g, g') = (g, g') \]

\[ \psi(g, g') = (g, g'g) \]

\[ \psi^{-1}(g, g') = (g, g'^{-1}) \]

\[ (g, g', g') \rightarrow (g, g', g''g) \]