November 3, hour 24: Prime and maximal ideals

Read Along. Slick Sections 2.1-2.3

Riddle Along. $G(x) = ?$

Agenda. “Better ideals”.

... From now on, $R$ is commutative.

Maximal Ideals. 1. Definition.

2. If $R$ is maximal $\Rightarrow R/I$ is a field.

Example. $S = \{ x \in \mathbb{R}^+ \}$ $A_n = \{ a_i : a_n = 0 \}$

Fisby Theorem. Every ideal is contained in a maximal ideal.

Proof using Zorn’s Lemma.

Theorem. There exists a function

$$\lim : \left\{ {\text{bounded seqs}} \right\} \rightarrow \mathbb{R} \text{ s.t.}$$

1. If $(a_n)$ is convergent, $\lim a_n = \lim a_n$

2. $\lim (a_n + b_n) = \lim (a_n) + \lim (b_n)$ + more...

3. $\lim (a_n b_n) = \lim (a_n) \cdot \lim (b_n)$

Proof.

$$S = \{ \text{bounded seqs in } \mathbb{R} \}$$

$J$ - a maximal ideal containing $I$.

$$\lim : S \rightarrow S/J \cong \mathbb{R}$$
Prime Ideals. 1. Definition. \( P \subseteq R \) is prime if \( ab \in P \Rightarrow a \in P \) or \( b \in P \).

2. Theorem. \( R/P \) is a domain iff \( P \) is prime.

Proof. \( \Rightarrow \) \( ab \in P \Rightarrow [ab] = 0 \Rightarrow [a][b] = 0 \Rightarrow \exists \gamma \in \mathbb{Z} \) s.t. \( b \gamma \in P \).

\( \Leftarrow \) \( [a][b] = 0 \Rightarrow [ab] = 0 \Rightarrow ab \in P \Rightarrow a \gamma = 0 \Rightarrow b \gamma = 0 \).

Theorem. A maximal ideal is prime.

From this point, \( R \) is a Domain (no \( 0 \)-divisors).

Primes. 1. \( a \mid b \) \( (a \mid b \land b \mid a = \Rightarrow a \mid c b) \) don't 6.

2. \( \gcd(a, b) = q \) \( \Rightarrow \gcd(a, q) = q \) \( \Rightarrow q = uq \).

3. Primes: \( P \neq 0 \) non-unit \( \Rightarrow \) \( P \mid ab \Rightarrow P \mid a \) or \( P \mid b \).

4. Irreducible \( a \mid b \Rightarrow a \in R^* \lor b \in R^* \).

Claim. prime \( \Rightarrow \) irreducible.

Counterexample: in \( \mathbb{Z}[\sqrt{-5}] \), 2 is irreducible (for norm reasons) but not prime, as

\( p = a b \Rightarrow P \mid a = \Rightarrow a \in R^* \) \( \Rightarrow \) \( p \mid c b \Rightarrow cb = 1 \Rightarrow b \in R^* \) \( \exists (1-\sqrt{-5}) (1+\sqrt{-5}) = 6 \).