November 29 checkmarks

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12:08 PM

Theorem. $(R\text{-mod}, \oplus, \otimes, 0, R)$ is a "ring".

Theorem. $(M,N) \longmapsto M \otimes N$ is a "bifunctor".

Theorem. The universal property.

$$\begin{tikzcd} M \times N \arrow[r, "i", "\text{bilinear}"'] \arrow[dr, "\text{bilinear}"'] & M \otimes N \arrow[d, dashed, "\exists!"] \\ & P \end{tikzcd}$$

Example. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$ "Extension of scalars".
$\leftarrow$ a $\mathbb{Q}$-module.

In general, given $\phi : R \to S$ a ring morphism, S is an R module & set $M_S := S \otimes_R M$. Then M_S is an S-module and $R^n_S = S^n$.

Prop. For any domain R there is a unique field $Q(R)$ s.t.

"The field of fractions"

$$\begin{tikzcd} R \arrow[r, "1\text{-}1"] \arrow[dr] & Q(R) \arrow[d, dashed, "\exists!"] \\ & F \end{tikzcd}$$

Proof later.

<u>Claim</u> If M is torsion $\left[\forall m \in M\ \exists r \in R,\ r \neq 0,\ rm = 0\right]$ Then $M_{Q(R)} = 0$.

$a \otimes m = r\left(\frac{a}{r} \otimes m\right) = \frac{a}{r} \otimes rm = 0$

<u>Prop</u> If $M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i} \rangle$, then

1. $\dim_{Q(R)} M_{Q(R)} = k$
2. $\dim_{R/\langle p \rangle} M_{R/\langle p \rangle} = k + |\{i : p_i \sim p\}|$
3. $\dim_{R/\langle p \rangle} \operatorname{im}(m \mapsto p^s m)_{R/\langle p \rangle} = k + |\{i : p_i \sim p \ \&\ s < s_i\}|$

$$\operatorname{im}(m \mapsto p^s m) \overset{\text{as}}{\cong} \begin{cases} p^s R \cong R & \text{on } R \checkmark \\ R/\langle q^t\rangle & \text{on } R/\langle q^t\rangle \quad q \neq p \checkmark \\ 0 & \text{on } R/\langle p^t\rangle \quad s \geq t \checkmark \\ R/\langle p^{t-s}\rangle & \text{on } R/\langle p^t\rangle \quad s < t \checkmark \end{cases}$$

$R/\langle p^{t-s}\rangle \cong \operatorname{im} p^s$ on $R/\langle p^t\rangle$ via

$r \longmapsto p^s \cdot r$

$r \longleftarrow p^s r + p^t r'$

Localization & Fields of Fractions.

Let R be a commutative domain

Def A multiplicative subset S of $R \setminus \{0\}$. (contains 1, closed under $\times$)

Examples $R \setminus \{0\}$, $R \setminus P$ (P prime), Powers of $a \neq 0$.

Definition $S^{-1}R = \left\{\frac{r}{s}\right\} \Big/ \frac{r_1}{s_1} \sim \frac{r_2}{s_2}$ if $r_1 s_2 = r_2 s_1$ ✓

$\Big[\frac{r_1}{s_1} \sim \frac{r_2}{s_2},\ \frac{r_2}{s_2} \sim \frac{r_3}{s_3} \Rightarrow r_1 s_2 = r_2 s_1,\ r_2 s_3 = r_3 s_2 \Rightarrow$

$r_1 s_2 s_3 = r_2 s_1 s_3 = s_1 r_3 s_2 \Rightarrow r_1 s_3 = r_3 s_1 \Big]$ ✓

$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \ldots$ ✓

$\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \ldots$

$R \setminus \{0\}$ — "Field of Fractions $Q(R)$"

$R \setminus P$ — "localization at P"

$\{2^n\}$ — "dyadic rationals".

$R \to S^{-1}R$ is injective ✓

Abelian groups & the mult. groups of finite fields

$$A \cong \mathbb{Z}^k \oplus \bigoplus \mathbb{Z}/p_i^{s_i} \cong \mathbb{Z}^k \oplus \mathbb{Z}/a_1 \oplus \mathbb{Z}/a_2 \oplus \ldots \checkmark$$

$$a_1 \mid a_2 \mid a_3 \ldots$$

Theorem If F is finite, F^* is cyclic.

Proof Otherwise, $x^{a_1} - 1$ has too many roots. ✓