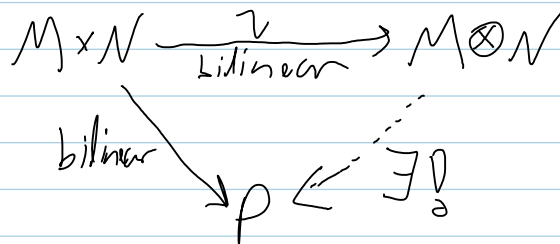


Theorem. $(R\text{-mod}, \oplus, \otimes, 0, R)$ is a "ring". ✓

Theorem. $(M, N) \mapsto M \otimes N$ is a "bifunctor". ✓

Theorem. The universal property. ✓



Example. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}^n \cong \mathbb{Q}^n$ "Extension of scalars". ✓
 \mathbb{Q} is a \mathbb{Q} -module.

In general, given $\phi: R \rightarrow S$ a ring morphism, S is an R module & set $M_S := S \otimes_R M$. Then M_S is an S -module and $R_S^n = S^n$. ✓

Prop. For any domain R there is a unique field $\mathbb{Q}(R)$ s.t. $R \xrightarrow{\iota} \mathbb{Q}(R)$ "The Field of Fractions". ✓
 $\downarrow \exists!$
 F Proof later.

Claim IF M is torsion $\left[\forall m \in M \exists r \in R \begin{matrix} r \neq 0 \\ rm = 0 \end{matrix} \right]$ then $M_{\mathbb{Q}(R)} = 0$. ✓
 $a \otimes m = r \left(\frac{a}{r} \otimes m \right) = \frac{a}{r} \otimes rm = 0$

Prop IF $M \cong R^k \oplus \bigoplus R/\langle p_i, s_i \rangle$, then

1. $\dim_{\mathbb{Q}(R)} M_{\mathbb{Q}(R)} = k$ ✓

2. $\dim_{R/\langle p \rangle} M_{R/\langle p \rangle} = k + |\{i : p_i \sim p\}|$ ✓

3. $\dim_{R/\langle p \rangle} \text{im}(M \rightarrow p^s M)_{R/\langle p \rangle} = k + |\{i : p_i \sim p \ \& \ s < s_i\}|$ ✓

$$\text{as } \text{im}(M \mapsto pM) \cong \begin{cases} p^s R \cong R \text{ on } R & \checkmark \\ R/\langle q^t \rangle \text{ on } R/\langle q^t \rangle & \text{qxp} \checkmark \\ 0 \text{ on } R/\langle p^t \rangle & \text{s} \neq t \checkmark \\ R/\langle p^t s \rangle \text{ on } R/\langle p^t \rangle & \text{sct} \checkmark \end{cases}$$

$$R/\langle p^t - s \rangle \cong \text{im } p^s \text{ on } R/\langle p^t \rangle$$

via

$$r \mapsto p^s \cdot r$$

$$r \longleftarrow p^s r + p^t r$$

Localization & Fields of fractions. Let R be a commutative domain

DEF A multiplicative subset S of $R \setminus \{0\}$. (contains 1, closed under \times)

Examples $R \setminus \{0\}$, $R \setminus P$ (P prime), Powers of $a \neq 0$.

Definition $S^{-1}R = \left\{ \frac{r}{s} \right\} / \frac{r_1}{s_1} \sim \frac{r_2}{s_2} \text{ if } r_1 s_2 = r_2 s_1$ ✓

$$\left[\frac{r_1}{s_1} \sim \frac{r_2}{s_2}, \frac{r_2}{s_2} \sim \frac{r_3}{s_3} \Rightarrow r_1 s_2 = r_2 s_1, r_2 s_3 = r_3 s_2 \Rightarrow \right.$$

$$r_1 s_2 s_3 = r_2 s_1 s_3 = s_1 r_3 s_2 \Rightarrow r_1 s_3 = r_3 s_1$$
 ✓

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \dots$$

$$\frac{r_1}{s_1} \cdot \frac{r_2}{s_2} = \dots$$

$R \setminus \{0\}$ - "Field of Fractions $\mathbb{Q}(R)$ "

$R \setminus P$ - "localization at P "

$R \rightarrow S^{-1}R$
is injective ✓

$\{2^n\}$ - "dyadic rationals".

Abelian groups & the mult. groups of finite fields

$$A \cong \mathbb{Z}^k \oplus \bigoplus \mathbb{Z}/p_i^{s_i} \cong \mathbb{Z}^k \oplus \mathbb{Z}/a_1 \oplus \mathbb{Z}/a_2 \oplus \dots \checkmark$$

$$a_1 | a_2 | a_3 \dots$$

Theorem If F is finite, F^* is cyclic.

Proof otherwise, $x^{a_1} - 1$ has too many roots. ✓