

↓ T2C2W: $[M \text{ f.g.}/R \text{ PID} \Rightarrow M \cong \overbrace{R^k \oplus \bigoplus R/\langle p_i^{s_i} \rangle}^{\text{unique}}]$
 \Rightarrow structure of f.g. Abelian groups, J.C.F.

Goal: The "ring" of modules.

Recall that $(R\text{-mod}, \oplus)$ is an "Abelian group" (really, an Abelian semi-group, and even this is not precise)

Tensor Products. Given M, N

$$M \otimes_R N := \left\{ \sum_{i=1}^n a_i (m_i \otimes n_i) : m_i \in M, n_i \in N, a_i \in R \right\} / \begin{array}{l} (am) \otimes n = a(m \otimes n) = m \otimes (an) \\ (m_1 + m_2) \otimes n = \dots \\ m \otimes (n_1 + n_2) = \dots \end{array}$$

$M \times N \xrightarrow{\text{bilinear}}$

Post Mortem: I should have defined $M \otimes_R N$ by its universal property.

2. I should have given the $\iota: F(X) \otimes F(Y) \rightarrow F(X \times Y)$

example. 1. Always injective? [not so easy!]

2. Isomorphism if X or Y are finite.

3. Not surjective if $R = \mathbb{Z}$, X, Y are infinite.

[not at all obvious!]

Example. $\dim V \otimes W = (\dim V) \cdot (\dim W)$

Example. If $q \in \gcd(a, b)$, $\frac{R}{\langle a \rangle} \otimes \frac{R}{\langle b \rangle} \cong \frac{R}{\langle q \rangle}$

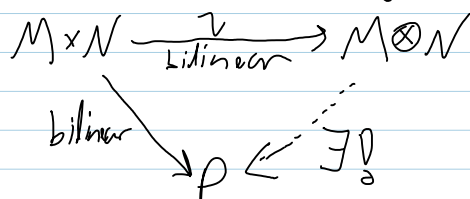
PF. $[r_1]_a \otimes [r_2]_b \rightarrow [r_1 \cdot r_2]_q$ $[q] \otimes [1] = [qa + qb] \otimes [1] = 0$
 $[r]_a \rightarrow [r]_a \otimes [1]_b$ $[r_1] \otimes [r_2] = [r_1] [r_2]$

done
line

Theorem. $(R\text{-mod}, \oplus, \otimes, 0, R)$ is a "ring".

Theorem. $(M, N) \mapsto M \otimes N$ is a "bifunctor".

Theorem. The universal property.



bilinear $\rightarrow \rho \leftarrow \exists!$