November 24, hour 31: The "ring" of modules

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10:59 PM

$\frac{1}{2}$ T2C2W: [M f.g./R PID $\Rightarrow M \cong R^k \oplus \overbrace{\bigoplus R/\langle p_i^{s_i}\rangle}^{\text{unique}}$]
$\Rightarrow$ structure of f.g. Abelian groups, J.C.F.

Goal: The "ring" of modules.

Recall that $(R\text{-mod}, \oplus)$ is an "Abelian group" (really, an Abelian semi-group, and even this is not precise)

Tensor Products. Given M, N

$$M \otimes_R N := \left\{ \sum_{i=1}^{n} a_i (m_i \otimes n_i) : n \in \mathbb{N}, a_i \in R \right\} \Big/ \begin{array}{l} (am)\otimes n = a(m\otimes n) = m\otimes(an) \\ (m_1+m_2)\otimes n = \ldots \\ m\otimes(n_1+n_2) = \ldots \end{array}$$

$M \times N \xrightarrow{\text{bilinear}}$

Post Mortem: 1. I should have defined $M \otimes_R N$ by its universal property.

2. I should have given the $\iota : F(X) \otimes F(Y) \to F(X \times Y)$ example.
1. Always injective! [not so easy!]
2. Isomorphism if X or Y are finite.
3. Not surjective if $R = \mathbb{Z}$, X, Y are infinite. [not at all obvious!]

Example. $\dim V \otimes W = (\dim V)\cdot(\dim W)$

Example. If $q \in \gcd(a,b)$, $q = sa + tb$, $\frac{R}{\langle a \rangle} \otimes \frac{R}{\langle b \rangle} \cong \frac{R}{\langle q \rangle}$

Pf. $[r_1]_a \otimes [r_2]_b \longrightarrow [r_1 \cdot r_2]_q$ $\qquad [q]\otimes[1] = [sa+tb]\otimes[1] = 0$

$[r]_q \longrightarrow [r]_a \otimes [1]_b$ $\qquad [r_1 r_2]\otimes[1] = [r_1][r_2]$

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done line

theorem. $(R\text{-mod}, \oplus, \otimes, 0, R)$ is a "ring".

theorem. $(M, N) \longmapsto M \otimes N$ is a "bifunctor".

Theorem. The universal property.

$$\begin{tikzcd} M \times N \arrow[r, "\iota"', "\text{bilinear}"] \arrow[dr, "\text{bilinear}"'] & M \otimes N \arrow[d, dashed, "\exists!"] \\ & P \end{tikzcd}$$

bilinear ρ $\exists!$