November 1, hours 22-23: Ideals, isomorphism theorems, prime and maximal ideals

October-28-11
11:10 PM

Agenda. Quotients, isomorphism theorems, “better rings.”

Read Along. Select 2.1-2.3.

HW 3 on web.

Def. $I \subset R$ is an ideal.

Claim. If $\phi : R \rightarrow S$ is a morphism of rings, then $\ker (\phi)$ is an ideal in $R$.

Q. Is every ideal a quotient?

Ans. Define $R/I$.

Example. $R[x]/\langle x^2 + 1 \rangle = \mathbb{R}.$

The Isomorphism theorems.

1. $\phi : R \rightarrow S \Rightarrow R/\ker (\phi) \cong \text{im } \phi.$

   (Example: $\text{ev}_c : R[x] \rightarrow \mathbb{C} \Rightarrow \mathbb{R} \cong \mathbb{C}$)

2. $A + I \subseteq \frac{A}{AI}$ $A \subset R$ subring, $I \subset R$ ideal

3. $I \subseteq R \Rightarrow \frac{R/I}{I} \cong R/I$

4. Given an ideal $I$ of $R$, there's a bijection between ideals $I \subset J \subset R$ and ideals of $R/I$.

Better Rings. 1. The ultimate:

   Field $[\text{commutative, } F \text{ of a group}]

   "division ring", if not commutative

   (Example: $H = \{a + b + wi + d k \} / i^2 = j^2 = k^2 = -1, ij = k$)

   useful for 3D rotations, etc...
2. (Integral) domains: Commutative, has no 0-divisors. How make? For ideals which, \( R/I \) is a field or a domain?

... From now on, \( R \) is commutative.

Maximal Ideals. 1. Definition.

2. \( I \subseteq R \) is maximal \( \iff \) \( R/I \) is a field.

Fishy proof: Use the 4th isomorphism theorem.

Honest proof: \( \Rightarrow \) \( x \not\in I \Rightarrow Rx + I = R \Rightarrow \exists y \in R \ yx + I = 1 + I \)

\( \iff \exists J \supseteq I, x \not\in J \Rightarrow \exists y \not\in J \ x = 1 + I \Rightarrow 1 + J \)

Examples. 1. \( \mathbb{Z} \) is a maximal ideal in \( \mathbb{Z} \).

2. \( S = \{a_n \in \mathbb{R} : \lim \ a_n = 0\} \)

Theorem. Every ideal is contained in a maximal ideal.

Proof using Zorn's Lemma.

**Theorem.** There exists a function

\[ \text{Lim}: \{\text{bdd seq's in } \mathbb{R}\} \to \mathbb{R} \quad \text{s.t.} \]

1. If \( (a_n) \) is convergent, \( \text{lim} a_n = \text{Lim} a_n \).

2. \( \text{Lim} (a_n + b_n) = \text{Lim} (a_n) + \text{Lim} (b_n) \) + more...

3. \( \text{Lim} (a_n b_n) = \text{Lim} (a_n) \cdot \text{Lim} (b_n) \)

Proof:

\[ S = \{\text{bdd seq's in } \mathbb{R}\} \quad I = \{a_n \in \mathbb{R} : \text{finlly ney n for any } n\}

J - a maximal ideal containing I.

\[ \text{Lim}: S \to S/J = \mathbb{R} \]
Prime Ideals. 1. Definition \( P \subset R \) is prime if \( ab \in P \implies a \in P \) or \( b \in P \).

2. Theorem. \( R/P \) is a domain iff \( P \) is prime.
   Proof: \( \implies \) \( ab \in P \implies [ab] = 0 \implies [a][b] = 0 \implies [a] = 0 \implies a \in P \)
   \( \implies \) \( ab \in P \implies [ab] = 0 \implies [a][b] = 0 \implies a \in P \) or \( b \in P \).

Theorem. A maximal ideal is prime.