\[ S = \{ \text{bounded sets in } \mathbb{R} \} \quad I = \{ (a_n) : \text{finitely many } a_i \neq 0 \} \]

Claim. \( J \) is maximal ideal containing \( I \).

**Definition.** Say that \( A \in \mathcal{N} \) is “essential” if \( 1_A \notin J \).

**Claim.** \( \{ A: A \text{ is essential} \} = \mu \) is a non-principal ultrafilter on \( \mathcal{N} \).

**Proof.** \( J \) is prime \( \Rightarrow \) \( (A \in \mathcal{N} \Rightarrow A \in \mu) \)

\( N \in \mathcal{N} \) because \( 1_N = 1_N \) is not in \( J \).

\[ A \in \mu \Leftrightarrow 1_A \notin J \Leftrightarrow (1_N - 1_A) \notin J \Leftrightarrow 1_A \in J \Leftrightarrow A \in \mu \]

Monotonically because \( J \) is an ideal: \( A \in \mu, B \in \mu \Rightarrow B \in \mu \Rightarrow 1_B \in J \Rightarrow 1_A = 1_B \cdot 1_A \in J \Rightarrow A \in \mu \).

Principally from the definition of \( I \).

**Definition.** \( \mathcal{J} = \{ (a_n) : \forall \varepsilon > 0 \exists N: |a_n| < \varepsilon \} \) is essential

**Claim.** \( J \subset \mathcal{J} \)

**Proof.** Suppose \( (a_n) \in J \), and \( \varepsilon > 0 \) is such that \( \{ N: |a_n| < \varepsilon \} \) is essential.

Let \( b_n = \sum_{n} a_n \varepsilon \) \( 1 \leq n \leq \varepsilon \), otherwise.
Then \( a_n b_n = 1 \) on an essential set, so \( a_n, b_n \neq 0 \), so \( a_n \neq 0 \) so \( a_n \not\in J = \emptyset \).

Now by the maximality of \( J \), \( J = \emptyset \).

Claim. For every \((a_n) \in S\) there is some \( x \in \mathbb{F} \) s.t. \( a_n - x \bar{T} \not\in J \)

(follows from convergence on ultrafilters)

\[ \Rightarrow \lim (a_n) = \lim (x \bar{T}) \]

Claim. The map \( \mathbb{R} \to S/\bar{J} \) via \( x \mapsto x \bar{T} \)

is injective and surjective.

Proof. Surjectivity was just shown. Injectivity is because any morphism of fields is injective, as field have no ideals to serve as kernels.

\[ \Rightarrow \text{using } x \mapsto x \bar{T} \text{ to identify } S/\bar{J} \text{ with } \mathbb{R}, \text{ the resulting } \lim \text{ has all the required properties.} \]