December 1, hour 34: Uniqueness, Corollaries

November-30-11
6:47 PM

Discuss the Final!

Goal. $M = R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle$. Uniqueness & corollaries.

Reminder. In a PID, $R/\langle a\rangle \otimes R/\langle b\rangle \cong R/\langle \gcd(a,b)\rangle$

Prop If $M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle$, then

1. $\dim_{Q(R)} M_{Q(R)} = k$
2. $\dim_{R/\langle p\rangle} M_{R/\langle p\rangle} = k + |\{i : p_i \sim p\}|$
3. $\dim_{R/\langle p\rangle} \operatorname{im}(m \mapsto p^s m)_{R/\langle p\rangle} = k + |\{i : p_i \sim p \;\&\; s < s_i\}|$

as
$$\operatorname{im}(m \mapsto p^s m) \cong \begin{cases} p^sR \cong R & \text{on } R \\ R/\langle q^t\rangle & \text{on } R/\langle q^t\rangle \quad q \not\sim p \\ 0 & \text{on } R/\langle p^t\rangle \quad s \geq t \\ R/\langle p^{t-s}\rangle & \text{on } R/\langle p^t\rangle \quad s < t \end{cases}$$

and
$$\ker(m \mapsto p^s m) \cong \begin{cases} 0 & \text{on } R \\ 0 & \text{on } R/\langle q^t\rangle \quad q \not\sim p \\ R/\langle p^t\rangle & \text{on } R/\langle p^t\rangle \quad s \geq t \\ R/\langle p^s\rangle & \text{on } R/\langle p^t\rangle \quad s < t \end{cases}$$
$R/\langle p^s\rangle \mapsto \ker$ by $[r]_{p^s} \mapsto [p^{t-s} r]_{p^t}$

So such a decomposition is unique! [Though not "canonical"] no do

$F[x]$ and the J.C.F. $T: V \to V$ makes V an $F[x] = R$ module, so $V \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle$. As $f(T) = 0$ for some f, $k = 0$. If F is alg. closed, $p_i = x - \lambda_i$

Q. What does $F[x]/(x-\lambda)^s$ look like as a vector space?

done line

Basis: $1, x-\lambda, (x-\lambda)^2, \ldots, (x-\lambda)^{s-1}$

$T - \lambda$ acts by "shift to the right" $\begin{pmatrix} 0 & 0 & \\ 1 & 0 & \\ 0 & 1 & \ddots \\ \vdots & & \end{pmatrix}$

So T acts by $\begin{pmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{pmatrix}$

Corollary 2. *Over an algebraically closed field* $\mathbb{F}$, *every square matrix* A *is conjugate to a block diagonal matrix* $B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$, *where each* B_i *is either a* 1×1 *matrix* (λ_1) *for some* $\lambda_i \in \mathbb{F}$, *or an* $s_i \times s_i$ *matrix with* λ_i*'s on the diagonals,* 1*'s right below the diagonal, and* 0*'s elsewhere,*

$$\begin{pmatrix} \lambda_i & 0 & \cdots & \cdots & 0 & 0 \\ 1 & \lambda_i & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & \lambda_i & 0 \\ 0 & 0 & \cdots & 0 & 1 & \lambda_i \end{pmatrix},$$

for some $\lambda_i \in \mathbb{F}$ *and for some* $s_i \geq 2$. *Furthermore,* B *is unique up to a permutation of its blocks* B_i.
(Corollary: good old diagonalization.)

Challenge.
Open all the boxes!

Find an algorithm to find B_j is it the same [at least when all λ_i's are different] as the one you learned in Junior high?