

**1 THEOREM, 2 COROLLARIES, 4 WEEKS**

**Theorem 1.** *Let  $M$  be a finitely generated module over a principal ideal domain  $R$ . Then*

$$M \cong R^k \oplus R/\langle p_1^{s_1} \rangle \oplus R/\langle p_2^{s_2} \rangle \oplus \cdots \oplus R/\langle p_n^{s_n} \rangle,$$

where  $k \geq 0$ ,  $p_1, \dots, p_n \in R$  are primes, and  $s_1, \dots, s_n$  are positive integers. Furthermore, up to a permutation of the  $R/\langle p_i^{s_i} \rangle$  factors, this decomposition is unique.

**Corollary 1.** *If  $A$  is a finitely generated Abelian group, then uniquely up to a permutation,*

$$A \cong \mathbb{Z}^k \oplus (\mathbb{Z}/p_1^{s_1}) \oplus \cdots \oplus (\mathbb{Z}/p_n^{s_n}).$$

(Corollaries:  $\mathbb{Z}^6 \not\cong \mathbb{Z}^7$ , the automorphism group of  $\mathbb{Z}/p$  is cyclic for any prime  $p$ .)

**Corollary 2.** *Over an algebraically closed field  $\mathbb{F}$ , every square matrix*

*$A$  is conjugate to a block diagonal matrix  $B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$ ,*

where each  $B_i$  is either a  $1 \times 1$  matrix ( $\lambda_i$ ) for some  $\lambda_i \in \mathbb{F}$ , or an  $s_i \times s_i$  matrix with  $\lambda_i$ 's on the diagonals, 1's right below the diagonal, and 0's elsewhere,

$$\begin{pmatrix} \lambda_i & 0 & \cdots & \cdots & 0 & 0 \\ 1 & \lambda_i & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & \lambda_i & 0 \\ 0 & 0 & \cdots & 0 & 1 & \lambda_i \end{pmatrix},$$

for some  $\lambda_i \in \mathbb{F}$  and for some  $s_i \geq 2$ . Furthermore,  $B$  is unique up to a permutation of its blocks  $B_i$ .

(Corollary: good old diagonalization.)

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<sup>1</sup><http://katlas.math.toronto.edu/drorbn/AcademicPensieve/Classes/11-1100/>. Created November 8, 2010, edited November 2, 2011.