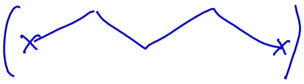
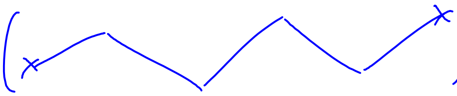
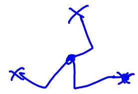
November 22, hour 29: Completely regular spaces

November-19-10
8:55 AM

Goal/Reminder. Compact & T_2 $\longrightarrow$ T_4; metric $\longrightarrow$ T_4 $\xrightarrow[\text{(apologies to his name)}]{\checkmark \text{ Urysohn}}$ $T_{4.5}$ $\xrightarrow{\checkmark}$ $T_{3.5}$ $\xrightarrow{\text{Today}}$ embeds in a cube I^A

+ Tietze, if time.

Riddle Along. $\mathcal{C}($  $) = ?$

$\mathcal{C}($  $) = ?$ $\quad \mathcal{C}($  $) = ?$

Read Along. 38(~), 35.

Challenge. Philosophically speaking, there should be a route from "metric" to "embeds in I^A" not going through Urysohn. Find It!

Proposition. X is $T_{3.5}$ iff $\{[f \neq 0] : f : X \to \mathbb{R} \text{ cont.}\}$ is a basis for the topology of X.

Claim. 1. If X is $T_{3.5}$ & $Y \subset X$, then Y is $T_{3.5}$

2. If X_α is $T_{3.5}$ $\forall \alpha$, then so is $\prod X_\alpha$

Given X let $C_X = C(X, I) = \left\{ \substack{\text{cont.} \\ f : X \to I} \right\}$ and let

$\phi : X \longrightarrow I^{C_X}$ be $\prod_f f$; $\phi_f = f$; ; $\phi(x)_f = f(x)$.

Theorem. ϕ is an embedding iff X is $T_{3.5}$.

(def: embedding: homeomorphism into its image).

Proof. $\Rightarrow$ A subspace of $T_{3.5}$ is $T_{3.5}$.

$\Leftarrow$ ϕ is clearly 1-1. If $U \subset X$ is $\overset{\text{basic}}{\text{open}}$, we need to show that $\phi(U)$ is open in $\phi(X)$. Indeed,

$$\phi([f \neq 0]) = \phi(X) \cap [\pi_f \neq 0]$$

done line.

Tietze's Theorem. If X is T_4, $A \subset X$ closed, and $f : A \to \mathbb{R}$ is cont., then there is a cont. extension $\tilde{f}$

of f to all of X. (So $\tilde{f}|_A = f$)

Remark. Tietze $\Rightarrow$ Urysohn.

Pf. of Tietze. Assume first that f is bndd.

<u>Lemma</u> Suppose $\tilde{f}_0 : X \to \mathbb{R}$ is such that $|f - \tilde{f}_0| \leqslant \epsilon$ on A; then $\exists\, \tilde{f}_1 : X \to A$ s.t. $|\tilde{f}_0 - f_1| \leqslant \epsilon$ everywhere & $|f - \tilde{f}_1| \leqslant \frac{2\epsilon}{3}$ on A.

So construct $\tilde{f}_0 = 0, \tilde{f}_1, \tilde{f}_2 \ldots$ s.t.

$$|f - \tilde{f}_n| < \left(\tfrac{2}{3}\right)^n M \quad \text{on } A$$

$$|\tilde{f}_n - \tilde{f}_{n+1}| < \left(\tfrac{2}{3}\right)^{n-1} \quad \text{on } X$$

let $\tilde{f}(x) = \lim_{n\to} \tilde{f}_n(x)$ (limit exists as $\tilde{f}_n(x)$ is cauchy)

<u>Lemma</u> A uniformly convergent sequence of cont. functions converges to a cont. limit.

<u>Proof</u> ....

Now do the unbounded case.....