

First Isomorphism Theorem. Given  $\phi: G \rightarrow H$ ,  
 $G/\ker \phi = \text{im } \phi$ . (Following Selick)

Proposition If  $H, K < G$  then  $HK = KH \iff HK < G$ .

Corollary  $H, K < G, H < N_G(K) \implies HK < G$  &  $K \triangleleft HK$ .

Corollary  $K \triangleleft G, H < G \implies HK < G$ . Should have used "N" for the normal one

Second Isomorphism Theorem.  $H, K < G, H < N_G(K)$

$$\implies H \cap K \triangleleft H, K \triangleleft HK \text{ and } \frac{HK}{K} = \frac{H}{H \cap K}$$

Proof.  $H \cap K \triangleleft H$ : Obvious.

I'm missing a good example!

$HK$  is a subgroup:  $h_1 k_1 h_2 k_2 = h_1 h_2 h_2^{-1} k_1 h_2 k_2 = h_1 h_2 k_1' k_2$ .

$$(h_1 k_1)^{-1} = k_1^{-1} h_1^{-1} = h_1^{-1} h_1 k_1^{-1} h_1^{-1} = h_1^{-1} k_1'$$

$$K \triangleleft HK: K^{-1} h^{-1} k_1 h k = K^{-1} k_1' k$$

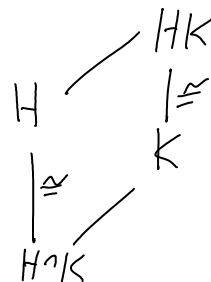
Now the isomorphism:

$$\phi: h(H \cap K) \mapsto hK \text{ clearly well defined.}$$

$$\psi: hK \mapsto h(H \cap K)$$

$$h_1 k_1 K = h_2 k_2 K \iff h_1 K = h_2 K \iff h_1 h_2^{-1} \in K \cap H.$$

In a diagram:



Third Isomorphism Theorem.  $K \triangleleft G, H \triangleleft G, K < H$

$$\implies H/K \triangleleft G/K \text{ and } (G/K)/(H/K) \cong G/H.$$

Fourth Isomorphism Theorem. If  $N \triangleleft G$  there's a bijection between subgroups of  $G/N$  and subgroups of  $G$  that contain  $N$ . This

subgroups of  $G$  that contain  $N$ . This bijection preserves inclusions, indices, intersections, and normality of inclusions.

The Butterfly.  $1 < a \triangleleft A < G, 1 < b \triangleleft B < G$   
 $\Rightarrow a(A \cap b) \triangleleft a(A \cap B),$  (Following Lang)

$$(a \cap B)b \triangleleft (A \cap B)b,$$

$$\text{and } a(A \cap B)/a(A \cap b) \cong (A \cap B)b/(a \cap B)b$$

"The B quotient in the A scale"  $\cong$  "The A quotient in the B scale"

Proof. Normality is obvious.

$$(A \cap B) \cdot a(A \cap b) = a(A \cap b)(A \cap B) = a(A \cap B), \text{ so}$$

$$a(A \cap B)/a(A \cap b) = (A \cap B)a(A \cap b)/a(A \cap b) \cong$$

$$\cong A \cap B / (A \cap B) \cap a(A \cap b) = \underbrace{(A \cap B) / (a \cap B)b \cap a(A \cap b)}_{\text{symmetric.}}$$

$$\text{as } (A \cap B) \cap a(A \cap b) = \underbrace{(a \cap B)b \cap a(A \cap b)}_{\text{"B" } \cap a(A \cap b)} \xrightarrow{\text{sym } C} \downarrow$$

$$\text{Given } \alpha \in B, \alpha \in a, \beta \in A \cap b, \alpha \beta \in B \Rightarrow \alpha \in B \Rightarrow \alpha \beta \in (a \cap B) \cap b$$

Jordan-Hölder Now follows....

The vector space analog.

Iso 2:

$$\frac{C+D}{D} \cong C/C \cap D$$

$$a+(A \cap B)/a+(A \cap b) \cong b+(B \cap A)/b+(B \cap a)$$

as both are

$$A \cap B / (a \cap B) + b \cap a + (A \cap b)$$

$$(A \cap B) \cap (a + A \cap b) = (a \cap B + b) \cap (a + A \cap b) \supseteq \checkmark$$

c ✓