(Following Selick)

First Isomorphism Theorem. Given $\phi: G \to H$,

$$G/\ker\phi = \operatorname{im}\phi.$$

<u>Proposition</u> If $H, K < G$ then $HK = KH \iff HK < G$.

<u>Corollary</u> $H, K < G$, $H < N_G(K) \Rightarrow HK < G$ & $K \triangleleft HK$.

<u>Corollary</u> $K \triangleleft G, H < G \Rightarrow HK < G$.

[Should have used "N" for the "normal" one]

Second Isomorphism Theorem. $H, K < G$, $H < N_G(K)$

$\Rightarrow H \cap K \triangleleft H$, $K \triangleleft HK$ and $HK/K = H/H\cap K$

I'm missing a good example!

Proof. $H \cap K \triangleleft H$: obvious.

HK is a subgroup: $h_1k_1h_2k_2 = h_1h_2h_2^{-1}k_1h_2k_2 = h_1h_2k_1'k_2$.

$$(h_1k_1)^{-1} = k_1^{-1}h_1^{-1} = h_1^{-1}h_1k_1^{-1}h_1^{-1} = h_1^{-1}k'$$

$K \triangleleft HK$: $k^{-1}h^{-1}k_1hk = k^{-1}k_1'k$.

Now the isomorphism:

$\phi: h(H\cap K) \longmapsto hK$ clearly well defined.

$\psi: hkK \longmapsto h(H \cap K)$

$h_1k_1K = h_2k_2K$ iff $h_1K = h_2K$ iff $h_1h_2^{-1} \in K \cap H$.

In a diagram:

$$\begin{tikzcd} & HK \arrow[d, dash, "\cong"] \\ H \arrow[ur, dash] \arrow[d, dash, "\cong"'] & K \arrow[dl, dash] \\ H\cap K & \end{tikzcd}$$

Third Isomorphism Theorem. $K \triangleleft G, H \triangleleft G, K < H$

$\Rightarrow H/K \triangleleft G/K$ and $(G/K)/(H/K) \cong G/H$.

Fourth Isomorphism Theorem. If $N \triangleleft G$ there's a bijection between subgroups of G/N and subgroups of G that contain N. This subgroups of G that contain N. This bijection preserves inclusions, indices, intersections, and normality of inclusions.

The Butterfly. $1 < a \lhd A < G$, $1 < b \lhd B < G$

(Following Lang)

$$\Rightarrow \quad a(A\cap b) \lhd a(A\cap B),$$
$$(a\cap B)b \lhd (A\cap B)b,$$

and $\quad a(A\cap B)/a(A\cap b) \cong (A\cap B)b/(a\cap B)b$

"The B quotient in the A scale" $\cong$ "The A quotient in the B scale"

Proof. Normality is obvious.

$(A\cap B)\cdot a(A\cap b) = a(A\cap b)(A\cap B) = a(A\cap B)$, so

$$a(A\cap B)/a(A\cap b) = (A\cap B)a(A\cap b)/a(A\cap b) \cong$$
$$\cong A\cap B/(A\cap B)\cap a(A\cap b) = \underbrace{(A\cap B)/(a\cap B)b\cap a(A\cap b)}_{\text{symmetric.}}$$

as $(A\cap B)\cap a(A\cap b) = (a\cap B)b \cap a(A\cap b)$

$\quad \| \quad$ easy $\quad \subset$

$B\cap a(A\cap b)$

Given $\alpha\beta$, $\alpha\in a$, $\beta\in A\cap b$, $\alpha\beta\in B \Rightarrow \alpha\in B \Rightarrow \alpha\beta\in(a\cap B)\cap b$

Jordan-Hölder now follows....

The Vector space analog.

Iso 2: $\dfrac{C+D}{D} \cong C/_{C\cap D}$

$$a+(A\cap B)/a+(A\cap b) \cong b+(B\cap A)/b+(B\cap a)$$

as both are

$$A\cap B/(a\cap B)+b \cap a+(A\cap b)$$

$$(A\cap B)\cap(a+A\cap b) = (a\cap B+b)\cap(a+A\cap b)$$

$\supset \checkmark$

c √