September 30, hour 9: S_n and A_n

September-21-10
12:42 PM

Class photo, HW1 on web!

Read Along: Selick notes 1.6, Hungerford's book 7.10.

The Symmetric Group.

Def $(-1)^\sigma = \text{sign}(\sigma) = \prod_{i<j} \text{sign}(\sigma(j) - \sigma(i))$

Claim $(-1)^{\sigma\tau} = (-1)^\sigma (-1)^\tau$

Pf $(-1)^{\sigma\tau} = \prod_{i<j} \text{sign}(\sigma\tau(j) - \sigma\tau(i)) =$

$$\prod_{i<j} \text{sign}(\tau(j) - \tau(i)) \prod_{i<j} \frac{\text{sign}(\sigma\tau(j) - \sigma\tau(i))}{\text{sign}(\tau(j) - \tau(i))} =$$

$$= \text{sign}(\tau) \cdot \text{sign}(\sigma)$$

So $\text{sign}: S_n \to \{\pm 1\} = \mathbb{Z}/2$. Let $A_n = \ker \text{sign}$. "the alternating group"

[A_n is the set of perms that can be written as an even product of transpositions]

Theorem. A_n is simple for $n \neq 4$.

Cycle Decomposition. $(12)(345) = [21453] = 21453$

Claim If $\sigma = (a_1 \dots a_k)$ and $\tau = [\tau_1 \tau_2 \dots \tau_n]$, then

$$\sigma^\tau = \tau^{-1} \sigma \tau = (\tau^{-1}(a_1), \tau^{-1} a_2, \dots \quad)$$

Corollary σ is conjugate to σ' iff they have the same cycle lengths

Corollary $\#(\text{Conjugacy classes of } S_n) = P(n)$ done line

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Lemma 1. Every element of A_n is a product of 3-cycles.

Pf $(12)(23) = (123)$, $(123)(234) = (12)(34)$ ....

Lemma 2. If $N \triangleleft A_n$ contains a 3-cycle, then $N = A_n$

Pf WLOG, $(123) \in N$. Claim For $\sigma \in S_n$, $(123)^\sigma \in N$ $\left(\begin{matrix}\sigma \in A_n \checkmark \\ \sigma = (12)\sigma' \checkmark\end{matrix}\right)$

So N contains all 3-cycles ... $\square$

Now take $N \triangleleft A_n$ w/ $N \neq \{1\}$ [Always take the commutation w/ a 3-cycle]

Case 1. N contains an element w/ cycle of length ≥ 4

$\sigma = (123456)\sigma' \in N$ $\quad \sigma^{-1}(123)\sigma(123)^{-1} = (136)$

Case 2. N contains an element $\sigma = (123)(456)\sigma'$

consider $\sigma^{-1}(124)\sigma(124)^{-1}$

Case 3. N contains $\sigma = (123)$ (product of pairs)

Then $\sigma^2 = (132)$ ...

Case 4. Every element of N is a product of disjoint 2-cycles.

$\sigma = (12)(34)\sigma' \Rightarrow \sigma^{-1}(123)\sigma(123)^{-1} = (13)(24) = \tau \in N$

$\Rightarrow \tau^{-1}(125)\tau(125)^{-1} = (13452) \in N$