Red Along: Def: 4.5, Section 1.8.
A & B play as follows:
1. A writes 1...18 on the faces of 3 blank dice.
2. B chooses one, then A chooses another.
3. They play “War” a 1000 times, on money.
Whom would you rather be?

Def: G finite, a p-group, a Sylow-p group, Syl_p(G)

Theorem: 1. Sylow p-groups always exist; Syl_p(G) ≠ Ø.
2. Every p-group is contained in a Sylow-p group.
3. All Sylow-p subgroups of G are conjugate, and
   \[ |N_p(G)| = |Syl_p(G)| = 1 \mod p. \land |N_p(G)| \mid |G| \]

Tools:
1. Every G-set is a union of orbits; each one has order dividing the order of |G|.
2. Cauchy’s Theorem. In an Abelian group of order divisible by p, there’s an element of order p.

From: Pick x \in G. If p | |G|, done. Else p \nmid |G|, so \exists y \in G with
   \[ |y| = p \] in \ G \leq G. So y^p = e. Write y = pζ + r, with 0 ≤ r < p.
   Get: e = y^{pζ} = x * y^r \Rightarrow y^r \leq G \Rightarrow r = 0, so p | |y|.

3. If a p-element x, or a p-group H, normalizes P \leq Syl_p(G),
then x \in P, or H \leq P.

Step 1. Syl_p(G) ≠ Ø

Step 2. If P \leq Syl_p(G), \[ |\text{conjugates of } P| = 1 \mod p. \]

Step 3. If P \leq Syl_p(G) \land H \leq G is a p-group, then H
   is contained in a conjugate of P (and the rest follows)
Semi-Direct Products. If $N \triangleleft G$, $H \leq G$, consider $H \times N$ with $HN$.

There's always $\phi : H \times N \to HN$ by $(h, n) \mapsto hn$.

In general, nothing to say. $\text{Im } \phi$ might not be a group.

If $N \cap H = \{e\}$, injective, might not be surjective.

If $N \cap H = \{e\}$ & $N \triangleleft G$ & $H \triangleleft G$, then $[N, H] = \{e\}$ & $HN \cong H \times N$.

The interesting case is when $N \cap H = \{e\}$, $N \triangleleft G$, $H$ not.

Get $H \cong \text{Aut}(N)$ by $h \mapsto (n \mapsto n^h = h^{-1}nh)$

or $\phi_1(n) = h^*nh$

$h_1 n_1 h_2 n_2 = h_1 h_2 \phi_1(h_2 (n_2)) n_2$

Definition. Given abstract $N, H$ & $\phi : H \to \text{Aut}(N)$, the semi-direct product $HN$. ...

Prop. 1. In the above case, $\phi : H \times N \to HN$ is an isomorphism.

2. $N \triangleleft (HN)$ and $HN / N \cong H$. 