November 4, hour 24: Maximal & Prime Ideals

November-03-10
2:35 PM

No Tuesday class next week!

Read Along. Selick 2.2, 2.3.

Goal. "better rings" and their ideals, some humility.

Reminders. All rings [today] are commutative; Fields ($x \neq 0 \Rightarrow \exists x^{-1}$), domains (no 0-divisors).

Maximal Ideals. 1. Definition.

2. $I \subset R$ is maximal $\iff R/I$ is a field.

Fishy proof: Use the 4th isomorphism theorem.

Honest proof: $\Rightarrow$: $x \notin I \Rightarrow Rx + I = R \Rightarrow \exists y \in R \;\; yx + I = 1 + I$

$\Leftarrow$ $J \supsetneq I$, $x \in J \setminus I \Rightarrow [x]_I \neq 0 \Rightarrow \exists y \;\; xy - 1 \in I \Rightarrow 1 \in J$

Examples. 1. $p\mathbb{Z}$ is a maximal ideal in $\mathbb{Z}$.

2. $S = \ell^\infty = \left\{ \begin{smallmatrix}\text{bndd seq's}\\ \text{in } \mathbb{R}\end{smallmatrix} \right\}$ $\quad A_n = \{(a_i) : a_n = 0\}$

Theorem. Every ideal is contained in a maximal ideal.

Proof using Zorn's Lemma.

Theorem There exists a function

$\operatorname{Lim} : \left\{ \begin{smallmatrix}\text{bndd seq's}\\ \text{in } \mathbb{R}\end{smallmatrix} \right\} \to \mathbb{R}$ s.t.

1. If (a_n) is convergent, $\lim a_n = \operatorname{Lim} a_n$.
2. $\operatorname{Lim}(a_n + b_n) = \operatorname{Lim}(a) + \operatorname{Lim}(b_n)$
3. $\operatorname{Lim}(a_n b_n) = \operatorname{Lim}(a_n) \cdot \operatorname{Lim}(b_n)$

+ more....

Proof. $S = \{\text{bndd seq's in } \mathbb{R}\}$ $\quad I = \{(a_n) : a_n \neq 0 \text{ for finitely many } n\text{'s}\}$

J - a maximal ideal containing I.

Lin: $S \longrightarrow S/J = \mathbb{R}$

done line

Prime Ideals. 1. Definition $P \subset R$ is prime if $ab \in P$
$\Rightarrow a \in P$ or $b \in P$.

2. Theorem. R/P is a domain iff P is prime.

Proof. $\Rightarrow$ $ab \in P \Rightarrow [ab] = 0 \Rightarrow [a][b] = 0 \Rightarrow$ $[a]=0 \Rightarrow a \in P$ or $[b]=0 \Rightarrow b \in P$.

$\Leftarrow$ $[a][b]=0 \Rightarrow [ab]=0 \Rightarrow ab \in P \Rightarrow$ $a \in P \Rightarrow [a]=0$ or $b \in P \Rightarrow [b]=0$

Theorem. A maximal ideal is prime.