November 23, hours 29-30: Existence part for modules over a PID, justice for tensor products

November-19-10
11:23 AM

Construction notes. 1. Fast & informal reduction to a question about row- & column-reduction of matrices.

2. Fast & informal r/c-reduction over a Euc-Dom.

3. The proof, for real.

4. Justice for tensor products.

$ax+by=0 \qquad q=\gcd(a,b)=sa+tb$

$\Downarrow$

$a(sx'-\frac{b}{q}y')+b(tx'+\frac{a}{q})y' \qquad \begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}s & -b/q\\ t & a/q\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix} \qquad \begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}a/q & b/q\\ -t & s\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$

$=qx'$

$r_1 = ax + \dots \qquad\qquad r_1' = sr_1 + tr_2$
$r_2 = bx + \dots \quad \rightarrow \quad r_2' = -\frac{b}{q}r_1 + \frac{a}{q}r_2$

IT2C3W: $[M \text{ f.g.}/R \text{ PID} \Rightarrow M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle]$

$\Rightarrow$ structure of f.g. Abelian groups, J.C.F.

Theorem. If M is a finitely generated module over a PID R then M is isomorphic to

$M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle \qquad p_i$ prime, $s_i \in \mathbb{N}$.

Proof.

Pass 1. Rough, only for Euc. domains, ignoring infinities, stressing Gaussian elimination.

Pass 2. Use Kuperberg's trick & the $\begin{pmatrix}s & -b/q\\ t & a/q\end{pmatrix}$ matrices.

Post mortem note. I should have spent more time on presentation matrices for modules: what are they, when are two equivalent, in what sense infinite ones are allowed, the $T: V \to V / F[x]$ example.

Justice for tensor products. Given M, N

$$M \otimes_R N := \left\{ \sum_{i=1}^{n} a_i (m_i \otimes n_i) : n \in \mathbb{N}, a_i \in R \right\} \Big/ \begin{array}{l}(am)\otimes n = a(m\otimes n) = m\otimes(an)\\ (m_1+m_2)\otimes n = \dots\\ m\otimes(n_1+n_2) = \dots\end{array}$$

$M\times N \nearrow$ bilinear

Example. $\dim V\otimes W = (\dim V)\cdot(\dim W)$

Example. If $q \in \gcd(a,b)$, $q = sa+tb$, $\frac{R}{\langle a\rangle}\otimes\frac{R}{\langle b\rangle}\cong\frac{R}{\langle q\rangle}$

pf. $[r_1]_a\otimes[r_2]_b \longrightarrow [r_1\cdot r_2]_q$ $\qquad [q]\otimes[1] = [sa+tb]\otimes[1] = 0$

$[r]_q \longrightarrow [r]_a\otimes[1]_b$ $\qquad [r_1 r_2]\otimes[1] = [r_1][r_2]$

done line.

theorem. $(R\text{-mod}, \oplus, \otimes)$ is a "ring".

theorem. $(M,N)\longmapsto M\otimes N$ is a "bifunctor".