November 2, hours 22-23: Rings

October-28-10
11:51 AM

Read Along. Selick 2.1-23

HW. HW2 will be returned at end. HW3 on web by midnight.

From an email message I sent to a student, after (s)he inquired about his/her low mark:

Indeed your mark is low. It counts for just 25% of the total so in theory, you can still end this course with a pretty high grade. It is not for me to tell if this theory can become practice - I only know that your mark on this test was low, and I cannot speculate what the reason was. Bad luck? Insufficient preparation? Insufficient background? Bad day? Something else? Only you can know (and perhaps even you can't) if this was a one-time issue and you can expect things to get better, or if it is a good predictor for your final grade.

You definitely have to consider it as a serious warning sign, and you do have to figure out what went wrong and how it can be fixed.

Term test. Discussion at end.

Goal. 1. Rings, ideals, isomorphisms.

2. Prime & maximal ideals, domains and fields.

Definition 2.1.1. *A* ***ring*** *consists of a set R together with binary operations $+$ and $\cdot$ satisfying:*

1. $(R,+)$ *forms an abelian group,*
2. $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ $\forall a,b,c\in R$,
3. $\exists 1\neq 0\in R$ *such that* $a\cdot 1 = 1\cdot a = a$ $\forall a\in R$, *and*
4. $a\cdot(b+c) = a\cdot b + a\cdot c$ *and* $(a+b)\cdot c = a\cdot c + b\cdot c$ $\forall a,b,c\in R$.

Also define: Commutative ring.

Examples. $\mathbb{Z}$, $R[x]$, $M_{n\times n}(R)$

Morphisms, (Examples: 1. $\mathbb{Z}\longrightarrow\mathbb{Z}/n$
2. $R\longrightarrow R[x]$ at deg 0
3. $R\to M_{n\times n}(R)$ as diag
4. $ev_u: R[x]\to R$ (if R is commutative)
5. $M_{n\times n}(R[x])\cong M_{n\times n}(R)[x]$)

im, subring, ker, ideal.

Q. Is every ideal a quotient?

Ans. Define R/I.

The Isomorphism theorems. 1. $f:R\longrightarrow S \Rightarrow R/\ker(f) = \operatorname{im} f$.

2. $\dfrac{A+I}{I}\cong \dfrac{A}{A\cap I}$ $A\subset R$ subring, $I\subset R$ ideal.

3. $I\subset J\subset R$ ideals $\Rightarrow \dfrac{R/I}{J/I}\cong R/J$

4. Given an ideal I of R, there's a bijection between

ideals $I \subset J \subset R$ & ideals of R/I.

Better Rings. 1. The ultimate:

Field [commutative, $F\setminus 0$ a group]

[almost all of high-school & freshman algebra carries through]

("division ring", if not commutative

Example: $\mathbb{H} = \{a+bi+cj+dk\} / \begin{matrix} i^2=j^2=k^2=-1 \\ ij=k \end{matrix}$

useful for 3D rotations, etc...)

2. (Integral) domains: commutative, has no 0-divisors.

How make? For ideals which, R/I is a field or a domain?

.... from now on, R is commutative.

Maximal Ideals. 1. Definition.

2. Fishy existence

3. $I \subset R$ is maximal $\Longleftrightarrow$ R/I is a field.

Fishy proof: Use the 4th isomorphism theorem.

mostly done

Honest proof: $\Rightarrow$: $x \notin I \Rightarrow Rx + I = R \Rightarrow \exists y \in R \;\; yx + I = 1 + I$

$\Leftarrow$ $J \supsetneq I$, $x \in J \setminus I \Rightarrow [x]_I \neq 0 \Rightarrow \exists y \;\; xy - 1 \in I \Rightarrow 1 \in J$

Prime Ideals. 1. Definition $P \subset R$ is prime if $ab \in P$

$\Rightarrow a \in P$ or $b \in P$.

not done

2. Theorem. R/P is a domain iff P is prime.

Proof $\Rightarrow$ $ab \in P \Rightarrow [ab] = 0 \Rightarrow [a][b] = 0 \Rightarrow \begin{matrix} [a]=0 \Rightarrow a \in P \\ \text{or} \\ [b]=0 \Rightarrow b \in P. \end{matrix}$

$\Leftarrow$ $[a][b] = 0 \Rightarrow [ab] = 0 \Rightarrow ab \in P \Rightarrow \begin{matrix} a \in P \Rightarrow [a]=0 \\ \text{or} \\ b \in P \Rightarrow [b]=0 \end{matrix}$

Theorem. A maximal ideal is prime.