From an email message I sent to a student, after (s)he inquired about his/her low mark:

Indeed your mark is low. It counts for just 25% of the total so in theory, you can still end this course with a pretty high grade. It is not for me to tell if this theory can become practice - I only know that your mark on this test was low, and I cannot speculate what the reason was. Bad luck? Insufficient preparation? Insufficient background? Bad day? Something else? Only you can know (and perhaps even you can’t) if this was a one-time issue and you can expect things to get better, or if it is a good predictor for your final grade.

You definitely have to consider it as a serious warning sign, and you do have to figure out what went wrong and how it can be fixed.

Read Along, Serres 2.1-23
HW 1: HW 2 will be returned at HW end. HWs on web by midnight.

Term test. Discussion at end.

Goal. 1. Rings, ideals, isomorphisms.

2. Prime & maximal ideals, domains and fields.

Definition 2.1.1. A ring consists of a set $R$ together with binary operations $+$ and $\cdot$ satisfying:

1. $(R, +)$ forms an abelian group,

2. $(a \cdot b) \cdot c = a \cdot (b \cdot c) \ \forall a, b, c \in R$,

3. $\exists 1 \neq 0 \in R$ such that $a \cdot 1 = 1 \cdot a = a \ \forall a \in R$, and

4. $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c \ \forall a, b, c \in R$.

Also define: Comutative ring.

Examples. $\mathbb{Z}, R[x], M_{n \times n}(R)$

Morphisms,

1. $\mathbb{Z} \rightarrow \mathbb{Z}/n$

2. $R \rightarrow R[x]$ at $\deg 0$ 4. $R \rightarrow R[x]$ 

( if $R$ is commutative)

3. $R \rightarrow M_{n \times n}(R)$ as $\deg$

S. $M_{n \times n}(R[x]) \cong M_{n \times n}(R)[x]$

Im, subring, ker, ideal.

Q. Is every ideal a quotient of $R$?

Ans. Define $R/I$.

The Isomorphism theorems. 1. $f: R \rightarrow S \Rightarrow R/\ker(f) \cong f(R)$.

2. $A + I \cong A/IA$ ACR subring, ICR ideal

3. $I \cap J \subseteq \text{ideals} \Rightarrow \frac{R/I \times R/J}{I}$

4. Given an ideal $I$ of $R$, there's a bijection between
ideals $I \triangleleft \mathcal{R}$ & ideals of $\mathcal{R}/I$.

Better Rings. 1. The ultimate:

- Field $[\text{commutative, } F \text{ of a group}]
  \begin{array}{l}
  \text{“division ring”, if not commutative} \\
  \text{Example: } H = \{a + bi + cj + dk \mid i^2 = j^2 = k^2 = 1, ij = k\}
\end{array}$

- useful for 3D rotations, etc...

2. (Integral) domains: commutative, has no 0-divisors.

How make? For ideals which, $\mathcal{R}/I$ is a field or a domain?

... from now on, $\mathcal{R}$ is commutative.

Maximal Ideals. 1. Definition.

2. Fishy existence.

3. $I \triangleleft \mathcal{R}$ is maximal $\iff$ $\mathcal{R}/I$ is a field.

Fishy proof: Use the 4th isomorphism theorem.

Honest proof: $\Rightarrow$: $x \not\in I \Rightarrow Rx + I = R \Rightarrow \exists y \in \mathcal{R}: yx + I = 1 + I$.

$\Leftarrow$: $J \subseteq I$, $x \in J \setminus I \Rightarrow \exists a + b = 1 \Rightarrow \exists y \in \mathcal{R}$ $\Rightarrow xy - 1 \in I \Rightarrow 1 \in J$.

Prime Ideals. 1. Definition $P \triangleleft \mathcal{R}$ is prime if $ab \in P$

$\Rightarrow a \in P$ or $b \in P$.

2. Theorem. $\mathcal{R}/P$ is a domain iff $P$ is prime.

Proof: $ab \in P \Rightarrow [ab] = 0 \Rightarrow [a][b] = 0 \Rightarrow a = 0 \lor b = 0 \Rightarrow aP = 0$ or $bP = 0$.

$\Leftarrow$: $[a][b] = 0 \Rightarrow [ab] = 0 \Rightarrow ab \not\in P \Rightarrow \exists a \not\in P$.

Theorem. A maximal ideal is prime.