November 18, hour 28: The ring of modules

November-17-10

2:05 PM

IT2C3W: $[M \text{ f.g.}/R \text{ PID} \Rightarrow M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle]$

$\Rightarrow$ structure of f.g. Abelian groups, J.C.F.

Today. The "ring" of modules.

Reminder. An R-module: "A vector space over a ring".

Examples. 1. V.S. over a field.

2. Abelian groups over $\mathbb{Z}$.

3. Given $T: V \to V$, V over $F[x]$.

4. Given ideal $I \subset R$, R/I over R.

5. Column vectors R^n over $M_{n\times n}$ (Left module R-mod)
row vectors $(R^n)^T$ over $M_{n\times n}$ (right module mod-R)

Def/Claim. R-mod & mod-R are categories.

Def/claim. Submodules, $\ker \phi$, $\operatorname{im} \phi$, M/N

Boring Theorems. 1. $\phi: M \to N \Rightarrow M/\ker\phi \cong \operatorname{im}\phi$

2. $A, B \subset M \Rightarrow \frac{A+B}{B} = \frac{A}{A\cap B}$

3 $A \subset B \subset M \Rightarrow \frac{M/A}{B/A} = M/B$

4. Also dull.

Direct sums. $M, N \Rightarrow M \oplus N$

$$\begin{tikzcd} M \arrow[dr] \arrow[drr, bend left] & & \\ & M\oplus N \arrow[r, dashed, "\exists!"] & P \\ N \arrow[ur] \arrow[urr, bend right] & & \end{tikzcd} \text{sum} \qquad \begin{tikzcd} & & M \\ P \arrow[r, dashed, "\exists!"] \arrow[urr, bend left] \arrow[drr, bend right] & M\oplus N \arrow[ur] \arrow[dr] & \\ & & N \end{tikzcd} \text{product}$$

$$\left(\begin{tikzcd} & M \arrow[dr] & \\ P \arrow[ur] \arrow[dr] \arrow[rr, dashed, "\exists!?"] & & M\oplus N \\ & N \arrow[ur] & \end{tikzcd} \text{sum} \qquad \begin{tikzcd} & M \arrow[dr] & \\ M\oplus N \arrow[ur] \arrow[dr] \arrow[rr, dashed, "\exists!"] & & P \\ & N \arrow[ur] & \end{tikzcd} \text{product}\right)$$

differ for infinite families!

Example: $\dim(V\oplus W) = \dim V + \dim W$.

Example: if $\gcd(a,b)=1$ $1 = sa+tb$ [e.g., if R is a PID]

$\frac{R}{\langle a\rangle} \oplus \frac{R}{\langle b\rangle} \cong \frac{R}{\langle ab\rangle}$ via

$$\begin{tikzcd} R/\langle a\rangle \arrow[dr, "t\cdot b"] & & R/\langle a\rangle \\ \oplus & R/\langle ab\rangle \arrow[ur, "1"] \arrow[dr, "1"] & \oplus \\ R/\langle b\rangle \arrow[ur, "s\cdot a"'] & & R/\langle b\rangle \end{tikzcd}$$

$\mathbb{Z}/7 \oplus \mathbb{Z}/11 \oplus \mathbb{Z}/13 \cong \mathbb{Z}/77 \oplus \mathbb{Z}/13 \cong \mathbb{Z}/1{,}001$ "the chinese remainder theorem"

Tensor Products. Given M, N

$$M \otimes_R N := \left\{ \sum_{i=1}^{n} a_i (m_i \otimes n_i) : n \in \mathbb{N}, a_i \in R \right\} \Big/ \begin{array}{l} (am)\otimes n = a(m\otimes n) = m\otimes(an) \\ (m_1+m_2)\otimes n = \dots \\ m\otimes(n_1+n_2) = \dots \end{array}$$

$M \times N \nearrow$ bilinear

done line

Example. $\dim V \otimes W = (\dim V)\cdot(\dim W)$

Example. If $q \in \gcd(a,b)$, $q = sa + tb$, $\frac{R}{\langle a \rangle} \otimes \frac{R}{\langle b \rangle} \cong \frac{R}{\langle q \rangle}$

Pf. $[r_1]_a \otimes [r_2]_b \longrightarrow [r_1 \cdot r_2]_q$ $\qquad [q]\otimes[1] = [sa+tb]\otimes[1] = 0$

$[r]_q \longrightarrow [r]_a \otimes [1]_b$ $\qquad [r_1 r_2]\otimes[1] = [r_1][r_2]$

theorem. $(R\text{-mod}, \oplus, \otimes)$ is a "ring". } stated

theorem. $(M,N) \longmapsto M \otimes N$ is a "bifunctor".