

IT 2C3W: $[M \text{ f.g.} / R \text{ PID} \Rightarrow M \cong R^k \oplus \bigoplus R/\langle p_i^{j_i} \rangle]$

\Rightarrow structure of f.g. Abelian groups, J.C.F.

Today. The "ring" of modules.

Reminder. An R -module: "A vector space over a ring".

Examples. 1. V.S. over a field.

2. Abelian groups over \mathbb{Z} .

3. Given $T: V \rightarrow V$, V over $F[x]$.

4. Given ideal $I \subset R$, R/I over R .

5. Column vectors R^n over $M_{n \times n}$ (Left module R -mod)
 Row vectors $(R^n)^T$ over $M_{n \times n}$ (right module $\text{mod-}R$)

Def/claim. R -mod & $\text{mod-}R$ are categories.

Def/claim. Submodules, $\ker \phi$, $\text{im } \phi$, M/N

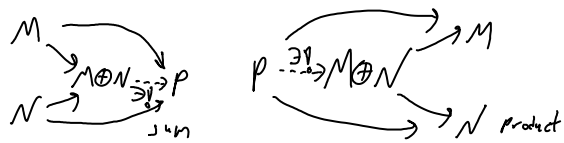
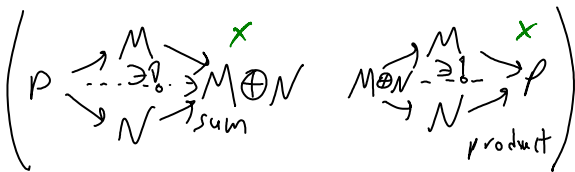
Boring Theorems. 1. $\phi: M \rightarrow N \Rightarrow M/\ker \phi \cong \text{im } \phi$

2. $A, B \subset M \Rightarrow \frac{A+B}{B} = \frac{A}{A \cap B}$

3. $A \subset B \subset M \Rightarrow \frac{M/A}{B/A} = M/B$

4. Also dual.

Direct sums. $M, N \Rightarrow M \oplus N$



differ for infinite families!

Example: $\dim(V \oplus W) = \dim V + \dim W$.

Example: if $\gcd(a,b)=1$ $1=sa+tb$ [e.g., if R is a PID]

$$\frac{R}{\langle a \rangle} \oplus \frac{R}{\langle b \rangle} \cong \frac{R}{\langle ab \rangle} \text{ via } \begin{array}{ccc} R/\langle a \rangle & \xrightarrow{t \cdot b} & R/\langle ab \rangle \xrightarrow{1} R/\langle a \rangle \\ \oplus & & \oplus \\ R/\langle b \rangle & \xrightarrow{s \cdot a} & R/\langle ab \rangle \xrightarrow{1} R/\langle b \rangle \end{array}$$

$\mathbb{Z}/7 \oplus \mathbb{Z}/11 \oplus \mathbb{Z}/13 \cong \mathbb{Z}/77 \oplus \mathbb{Z}/13 \cong \mathbb{Z}/1,001$ "the chinese remainder Theorem"

Tensor Products. Given M, N

$$M \otimes_R N := \left\{ \sum_{i=1}^n a_i (m_i \otimes n_i) : n \in \mathbb{N}, a_i \in R \right\} / \begin{array}{l} (am) \otimes n = a(m \otimes n) = m \otimes (an) \\ (m_1 + m_2) \otimes n = \dots \\ m \otimes (n_1 + n_2) = \dots \end{array}$$

$M \times N \nearrow$ bilinear

done LMI

Example. $\dim V \otimes W = (\dim V) \cdot (\dim W)$

Example. If $q \in \text{gcd}(a, b)$, $\frac{R}{\langle a \rangle} \otimes \frac{R}{\langle b \rangle} \cong \frac{R}{\langle q \rangle}$

pf. $[r_1]_a \otimes [r_2]_b \rightarrow [r_1 \cdot r_2]_q$ $[q] \otimes [1] = [q+tb] \otimes [1] = 0$
 $[r]_a \rightarrow [r]_a \otimes [1]_b$ $[r_1, r_2] \otimes [1] = [r_1][r_2]$

theorem. $(R\text{-mod}, \otimes, \otimes)$ is a "ring". } stated

theorem. $(M, N) \mapsto M \otimes N$ is a "bifunctor".