Noetherian Rings and Modules

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Following Selick's notes.

Definition An R-module M is Noetherian if every increasing sequence of submodules thereof is eventually constant. [Also: left Noetherian, right Noetherian, two-sided Noetherian, if R is not commutative]

Theorem TFAE:

1. M is Noetherian.
2. Every non-empty set of submodules of M contains a maximal element.
3. Every submodule of M is finitely generated.

Corollary Given $f: M \to N$ a morphism of R-modules, M is Noetherian iff both $\ker f$ and $\operatorname{im} f$ are.

Corollary If R is Noetherian then so is R/I.

The Hilbert Basis Theorem If R is commutative Noetherian, then so is $R[x]$.

Where do we use this first?

The Noetherian property is central in ring theory and in areas that make heavy use of rings, such as algebraic geometry. The reason behind this is that the Noetherian property is in some sense the ring-theoretic analogue of finiteness. For example, the fact that polynomial rings over a field are Noetherian allows one to prove that any infinite set of polynomial equations can be replaced with a finite set with the same solutions.

Pasted from <http://en.wikipedia.org/wiki/Noetherian_ring>

In other words, algebraic sets have a finite description.