# Finitely Generated Abelian Groups

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Following Lang, pp 42+

Given an Abelian group $A$, let $A(p)$ denote the set of elements of $A$ who order/period is a power of $p$.

**Theorem 1** If $A$ is torsion then it is a direct sum of its $A(p)$ subgroups.

**Proof** There's an obvious $\bigoplus_p A(p) \to A$. Injectivity: If $\sum x_p = 0$ in $A$, where $x_p \in A(p)$, then for any $q$, $x_q = \sum_{p \neq q} (-x_p)$. Comparing periods we come to $x_q = 0$. Surjectivity: If $x \in A$ is of period $m$, write $m = \prod p_i^{d_i}$ & $m_j = \prod_{i \neq j} p_i^{d_i}$. By the Chinese remainder theorem find $a_i$ with $\sum a_i m_i = 1$, and then $x = \sum a_i m_i x$. But for each $i$, $p_i^{d_i} a_i m_i x = 0$, so $a_i m_i x \in A(p)$. $\square$

**Definition** We say that a $p$-group is of type $(r_1, \dots r_s)$ if it is the product of cyclic groups of order $p^{r_i}$.

**Theorem 2** Every finite Abelian $p$-group is a product of cyclic $p$ groups. If it is of type $(r_1, \dots, r_s)$ where $r_1 \geq r_2 \geq \dots \geq r_s$, then the sequence $(r_i)$ is determined uniquely.

**Theorem 3** A finitely generated torsion free Abelian group is free.

**Theorem 4** If $A$ is a finitely generated Abelian group then $\mathrm{Tor}(A)$ is finite, $A/\mathrm{Tor}(A)$ is free, group then $\mathrm{Tor}(A)$ is finite, $A/\mathrm{Tor}(A)$ is free, and there exists a free $B < A$ s.t. $A = B \oplus \mathrm{Tor}(A)$.