December 2, hour 34: The Jordan Canonical Form

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The Best HW Ever: Open All boxes.

OTICIW: $[M$ f.g./R PID $\Rightarrow M \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle]$

$\Rightarrow$ structure of f.g. Abelian groups, J.C.F.

$F[x]$ and the J.C.F. $T: V \to V$ makes V an $F[x]=R$ module, so $V \cong R^k \oplus \bigoplus R/\langle p_i^{s_i}\rangle$. As $f(T)=0$ for some f, $k=0$. If F is alg. closed, $p_i = x-\lambda_i$

Q. What does $F[x]/(x-\lambda)^s$ look like as a vector space?

Basis: $1, x-\lambda, (x-\lambda)^2, \ldots, (x-\lambda)^{s-1}$

$T-\lambda$ acts by "shift to the right" $\begin{pmatrix} 0 & 0 & & \\ 1 & 0 & & \\ 0 & 1 & \ddots & \\ \vdots & & \ddots & \end{pmatrix}$

So T acts by $\begin{pmatrix} \lambda & & \\ 1 & \lambda & \\ & 1 & \lambda \end{pmatrix}$

Corollary 2. *Over an algebraically closed field $\mathbb{F}$, every square matrix A is conjugate to a block diagonal matrix* $B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$, *where each B_i is either a 1×1 matrix (λ_1) for some $\lambda_i \in \mathbb{F}$, or an $s_i \times s_i$ matrix with λ_i's on the diagonals, 1's right below the diagonal, and 0's elsewhere,*

$$\begin{pmatrix} \lambda_i & 0 & \cdots & \cdots & 0 & 0 \\ 1 & \lambda_i & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & \lambda_i & 0 \\ 0 & 0 & \cdots & 0 & 1 & \lambda_i \end{pmatrix},$$

for some $\lambda_i \in \mathbb{F}$ and for some $s_i \geq 2$. Furthermore, B is unique up to a permutation of its blocks B_i.
(Corollary: good old diagonalization.)

Challenge. Open all the boxes!

Find an algorithm to find B_i is it the same [at least when all λ_i's are different] as the one you learned in Junior high?

Cayley-Hamilton. Let R be any commutative ring, Det

$A \in M_{n\times n}(R)$, let $\chi_A(t) = \det(tI-A) \in R[t]$. Then $\chi_A(A)=0$.

Proof I. Substitute $t=A$, so

$\chi_A(A) = \det(A\cdot I - A) = \det(0) = 0.$

[$tr(tI-A) = nt - tr A$
so $nA - (tr A)I = 0$
so all matrices are diagonal?]

done line

Proof II. Recall that every matrix B has an "adjoint" B^* s.t. $B^*B = BB^* = \det(B)\cdot I$. Then

$$\underbrace{(tI-A)^*}_{=\sum B_k t^k}(tI-A) = \chi_A(t)I$$

as elements of $M_n R[t]$ & even $C_A[t]$, where $C_A = \{B : AB = BA\}$

There is a well-defined & multiplicative $ev_A : C_A[t] \to C_A[t]$. Applying to both sides, get

$$\left(\sum B_k A^k\right)\cdot(A-A) = \chi_A(A)I \qquad \square$$

Next Class:

1. Boxing day? ← chosen topic.
2. Odds & ends?
3. Something else?

   Alexander poly?