

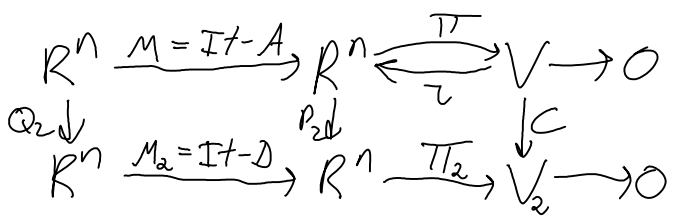
Boxing Day Handout, 2

December-07-10
6:38 AM

(P_2, Q_2) from C . Just take $P_2 = Q_2^{-1} = C$:

$$P_2 M Q_2^{-1} = C(I t - A) C^{-1} = I t - B = M_2$$

C from (P_2, Q_2) . Harder!



Take $C = \tau \circ P_2 \circ \tau^{-1}$ i.e., $C = L_D P_2$

Apply L_D to $(I t - D) Q_2 = P_2 (I t - A)$;

get $D(L_D P_2) = (L_D P_2) A$ \rightarrow 1. $C(I t - A) = (L_D P_2)(I t - A) = (I t - D) L_D P_2 = (I t - D) C$.

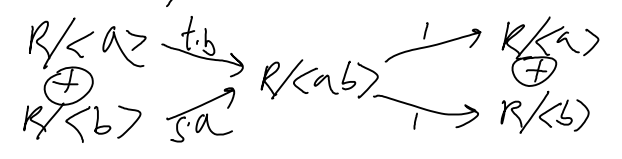
C is invertible with inverse $L_A P_2^{-1}$. Indeed,

\rightarrow 2. $\forall k D^k(L_D P_2) = (L_D P_2) A^k$, so for $Z \in M_n(F[t])$, $L_D(P_2 Z) = (L_D P_2)(L_A Z) = L_D(P_2 P_2^{-1}) = L_D(P_2) L_A(P_2^{-1}) = C \cdot L_A(P_2^{-1})$

$(P_1, Q_1) \leftrightarrow (P_2, Q_2)$ or more humbly, with $\Delta_n := \prod_{i=1}^n (t - \lambda_i)$

$$\begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 0 & \Delta_n \end{pmatrix} \leftrightarrow \begin{pmatrix} t - \lambda_1 & & \\ & \ddots & \\ & & t - \lambda_n \end{pmatrix}$$

We had. If $\gcd(a, b) = 1$ & $1 = s a + t b$, then $R/\langle a \rangle \oplus R/\langle b \rangle \cong R/\langle a, b \rangle$ via



Proof of Cayley-Hamilton.

Recall The adjoint matrix B^* ; $B^* B = B B^* = (\det B) I$. With $B = tI - A$, $B^* \in M_n(F[t]) = M_n(F)[t]$, and every coefficient of B^* commutes with A . Evaluate $B^* B = (\det B) I$ at $t = B$ & get

$$\begin{aligned}
 \text{ev}_B(B^* B) &= \text{ev}_B(B^*) \text{ev}_B(tI - B) = \\
 &= \text{ev}_B(B^*) (B - B) = 0 \\
 \text{ev}_B(\det B \cdot I) &= \chi_A(B) \cdot I
 \end{aligned}$$

$L_D: M_n(R) = M_n(F)[t] \rightarrow M_n(F)$ is "diff evaluation at D ":

$$L_D(\sum t^k Z_k) := \sum D^k Z_k$$

Matrix Version. $R^2 \xrightarrow{\begin{pmatrix} a & 1 \\ 0 & b \end{pmatrix}} R^2$



Note.

$$\frac{1}{\Delta_{n-1}(\lambda_1)} \left[\Delta_{n-1}(t) - \frac{\Delta_{n-1}(t) - \Delta_{n-1}(\lambda_1)}{t - \lambda_1} (t - \lambda_1) \right] = 1$$

Aside. $F^n \rightarrow \bigoplus R/\langle t - \lambda_i \rangle \cong R/\Delta_n$ is "Lagrange Interpolation"!