Dror Bar-Natan: Classes: 2010 Core Algebra:

Boxing Day Handout, 2

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6:38 AM

(P_2,Q_2) From C. Just take
$$P_2=Q_2^{-1}=C:$$
$$P_2MQ_2^{-1}=C(It-A)C^{-1}=It-D=M_2$$

C from (P_2,Q_2). Harder!

$$\begin{tikzcd}
R^n \arrow[r, "M=It-A"] \arrow[d, "Q_2"'] & R^n \arrow[r, shift left, "\pi"] \arrow[d, "P_2"'] & V \arrow[l, shift left, "\tau"] \arrow[r] \arrow[d, "C"] & 0 \\
R^n \arrow[r, "M_2=It-D"] & R^n \arrow[r, "\pi_2"] & V_2 \arrow[r] & 0
\end{tikzcd}$$

Take $C=\pi\circ P_2\circ\tau$; i.e., $C=L_DP_2$

Apply L_D to $(It-D)Q_2=P_2(It-A)$;

get $D(L_DP_2)=(L_DP_2)A$

1. $C(It-A)=(L_DP_2)(It-A)=(It-D)L_DP$
$=(It-D)C$.
2. $\forall k\ D^k(L_DP_2)=(L_DP_2)A^k$, so for $Z\in M_nF[t]$,
$L_D(P_2Z)=(L_DP_2)(L_AZ)$

C is invertible with inverse $L_AP_2^{-1}$. Indeed, $I=L_D(I)$
$$=L_D(P_2P_2^{-1})=L_D(P_2)L_A(P_2^{-1})=C\cdot L_A(P_2^{-1})$$

$(P_1,Q_1)\leftrightarrow(P_2,Q_2)$ or more humbly, with $\Delta_n:=\prod_{i=1}^n(t-\lambda_i)$,
$$\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & \Delta_n\end{pmatrix}\leftrightarrow\begin{pmatrix}t-\lambda_1 & & \\ & \ddots & \\ & & t-\lambda_n\end{pmatrix}$$

We had. If $\gcd(a,b)=1$ & $1=sa+tb$, then $R/\langle a\rangle\oplus R/\langle b\rangle\cong R/\langle ab\rangle$ via

$$\begin{tikzcd}
R/\langle a\rangle \arrow[dr, "t\cdot b"] & & R/\langle a\rangle \\
\oplus & R/\langle ab\rangle \arrow[ur, "1"] \arrow[dr, "1"'] & \oplus \\
R/\langle b\rangle \arrow[ur, "s\cdot a"'] & & R/\langle b\rangle
\end{tikzcd}$$

Proof of Cayley-Hamilton. Recall the adjoint matrix B^*; $B^*B=BB^*=(\det B)I$. With $B=tI-A$, $B^*\in M_n(F[t])=M_n(F)[t]$, and every coefficient of B^* commutes with A. Evaluate $B^*B=(\det B)I$ at $t=B$ & get
$$ev_B(B^*B)=ev_B(B^*)ev_B(tI-B)=$$
$$ev_B(B^*)(B-B)=0$$
$$ev_B(\det B\cdot I)=\chi_A(B)\cdot I$$

$L_D: M_n(R)=M_n(F)[t]\to M_n(F)$ is "left evaluation at D":
$$L_D\left(\sum t^kZ_k\right):=\sum D^kZ^k$$

Matrix Version.
$$\begin{tikzcd}
R^2 \arrow[r, "\begin{pmatrix}a&0\\0&b\end{pmatrix}"] \arrow[d, "\begin{pmatrix}a&-b\\t&s\end{pmatrix}"'] & R^2 \arrow[d, "\begin{pmatrix}1&-1\\tb&sa\end{pmatrix}"] \\
R^2 \arrow[r, "\begin{pmatrix}1&0\\0&ab\end{pmatrix}"] & R^2
\end{tikzcd}$$

Note.
$$\frac{1}{\Delta_{n-1}(\lambda_n)}\left[\Delta_{n-1}(t)-\frac{\Delta_{n-1}(t)-\Delta_{n-1}(\lambda_n)}{t-\lambda_n}(t-\lambda_n)\right]=1$$

Aside. $F^n\to\bigoplus R/\langle t-\lambda_i\rangle \xrightarrow{\sim} R/\Delta_n$ is "Lagrange Interpolation"!