Boxing Day

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The Setup. A an $n\times n$ matrix over alg. closed F, $B=C^{-1}AC$ its J.C.F.; V the corresponding $R=F[x]$ module, $P_{1,2}, Q_{1,2}\in M_{n\times n}(R)$ invertible, so that

$$\begin{tikzcd} R^n \arrow[r, "M"] & R^n \arrow[r] \arrow[d, "P_{1,2}"] & V \arrow[r] & 0 \\ R^n \arrow[u, "Q_{1,2}"] \arrow[r, "M_{1,2}"] & R^n \arrow[r] & V_{1,2} \arrow[r] & 0 \end{tikzcd} \qquad M=Ix-A$$

N_1 is the "prime factors" canonical form of V, $N_1=\begin{pmatrix}1 & & & & \\ & 1 & & & \\ & & p_1^{s_1} & & \\ & & & \ddots & \\ & & & & p_k^{s_k}\end{pmatrix}$ and N_2 is the "invariant factors" canonical form of V, $N_2=\begin{pmatrix}a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix}$ with $a_1 | a_2 | \dots | a_n$

Problem. Write (P_1,Q_1), (P_2,Q_2), C in terms of each other, first in the case where all eigenvalues are distinct, then in general.

Proof of $M=Ix-A$:

$$R^n \underset{h_1}{\overset{M}{\rightleftarrows}} R^n \underset{h_0}{\overset{\pi}{\rightleftarrows}} V \to 0$$

$\ker\pi\supset \operatorname{im} M$: easy

$H=0$: use the "homotopies"

$h_0(e_i)=e_i$

$h_1(x^k e_i)=\begin{cases}0 & k=0\\ & k>0.\end{cases}$

Need $h_0\circ\pi+M\circ h_1=I$; or

$M\circ h_1(x^k e_i)=(I-h_0\pi)(x^k e_i)=$

$=x^k e_i - A^k e_i=$

$=(x-A)\underbrace{(x^{k-1}+x^{k-2}A+x^{k-3}A^2+\dots+A^{k-1})e_i}$

(can also be done directly in fewer words...)

Assume all eigenvalues are different.

Given C, can take $Q_1=C$, $P_1=C^{-1}$!

Do the other way! $[\text{mod } C\leftrightarrow C^{-1}, Q\leftrightarrow Q^{-1}, P\leftrightarrow P^{-1}]$

$$\begin{tikzcd} R^n \arrow[r, "Ix-A"] \arrow[d, "Q"] & R^n \arrow[r] \arrow[d, "P"] & V \arrow[r] \arrow[d, dashed, "C"] & 0 \\ {} & {} & {} & \end{tikzcd}$$

$P=\sum x^i P_i$

$$\begin{array}{ccccccc} & & & & & & \\ Q\downarrow & & P\downarrow & & \downarrow C & & \\ \mathbb{R}^n & \xrightarrow{Ix-B} & \mathbb{R}^n & \longrightarrow & V & \longrightarrow & 0 \end{array}$$

$\Rightarrow C = \sum B^i P_i =: L_B P$

Given $Ix - B = P(Ix - A)Q^{-1}$ Expect $CAC^{-1} = B$

or $Qx - BQ = Px - PA$ or $CA = BC$

$CA = (L_B P)A$

$BC = B L_B P$ ✓

applying L_B:

$0 = B L_B P - L_B P A$

BTW, $C^{-1} = L_A P^{-1}$;

$\Rightarrow B^k (L_B P) = (L_B P) A^k \quad \forall k$

check:

$\Rightarrow L_B(PZ) = (L_B P) L_A(Z)$

$CC^{-1} = (L_B P)(L_A P^{-1}) =$

$= L_B(PP^{-1}) = L_B I = I$

Also, $P^{-1}x - P^{-1}B = xQ^{-1} - AQ^{-1}$

applying L_A, $A(L_A P^{-1}) = (L_A P^{-1})B$