Dror Bar-Natan: Classes: 2010-11: Math 1100 Core Algebra I:[1]

1 THEOREM, 2 COROLLARIES, 4 WEEKS TO GO

Theorem 1. *Let M be a finitely generated module over a principal ideal domain R. Then*

$$M \cong R^k \oplus R/\langle p_1^{s_1}\rangle \oplus R/\langle p_2^{s_2}\rangle \oplus \cdots \oplus R/\langle p_n^{s_n}\rangle,$$

where $k \geq 0$, $p_1, \ldots, p_n \in R$ are primes, and $s_1, \ldots, s_n$ are positive integers. Furtrhermore, up to a permutation of the $R/\langle p_i^{s_i}\rangle$ factors, this decomposition is unique.

Corollary 1. *If A is a finitely generated Abelian group, then uniquely up to a permutation,*

$$A \cong \mathbb{Z}^k \oplus (\mathbb{Z}/p_1^{s_1}) \oplus \cdots \oplus (\mathbb{Z}/p_n^{x_n}).$$

(Corollaries: $\mathbb{Z}^6 \ncong \mathbb{Z}^7$, the automorphism group of $\mathbb{Z}/p$ is cyclic for any prime p.)

Corollary 2. *Over an algebraically closed field $\mathbb{F}$, every square matrix A is conjugate to a block diagonal matrix* $B = \begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_n \end{pmatrix}$, *where each B_i is either a 1×1 matrix (λ_1) for some $\lambda_i \in \mathbb{F}$, or an $s_i \times s_i$ matrix with λ_i's on the diagonals, 1's right below the diagonal, and 0's elsewhere,*

$$\begin{pmatrix} \lambda_i & 0 & \cdots & \cdots & 0 & 0 \\ 1 & \lambda_i & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \ddots & \lambda_i & 0 \\ 0 & 0 & \cdots & 0 & 1 & \lambda_i \end{pmatrix},$$

for some $\lambda_i \in \mathbb{F}$ and for some $s_i \geq 2$. Furthermore, B is unique up to a permutation of its blocks B_i.
(Corollary: good old diagonalization.)

[1] `http://katlas.math.toronto.edu/drorbn/AcademicPensieve/Classes/10-1100/`. Produced November 8, 2010, edited November 9, 2010.