240Algebral-091029 Hour 22: Isomorphism of transformations, matrices

October-28-09
6:12 PM

At the end of the hour I will return the remaining exams!

VOLUNTEER NOTE - TAKERS
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Note again. It is completely pointless to upload notes to the class wiki without linking them anywhere!

"All" about one l.t.

Isomorphic:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\Phi"'] \arrow[dr, phantom, "?"] & W \arrow[d, "\Psi"] \\ V' \arrow[r, "T'"] & W' \end{tikzcd}$$

$\exists\, \Phi, \Psi$ isomorphisms of V.S., s.t. the diagram "commutes":

$$\Psi \circ T = T' \circ \Phi$$

Thm In finite dimension, $T: V \to W$ is "isomorphic" to $T': V' \to W'$ iff

$$(\dim V, \dim W, \operatorname{rank} T) = (\dim V', \dim W', \operatorname{rank} T')$$

Proof of $\Rightarrow$ Exercise; not very hard but not too easy

Proof of $\Leftarrow$ (sketch only): Choose bases with the following schematic behaviour:

$$(\overbrace{z_1 \dots z_k}^{\text{choose these first, in } N(T)}, \overbrace{u_1 \dots u_\ell}^{\text{then, extend with these}}) \text{ in } V \xrightarrow{T} (0, \dots, 0, \overbrace{w_1 = Tu_1, \dots, w_\ell = Tu_\ell}^{\text{these are forced}}, \overbrace{y_1 \dots y_n}^{\text{then, extend with these}}) \text{ in } W$$

$$\downarrow \Phi \qquad\qquad\qquad\qquad \downarrow \Psi$$

$$(z'_1 \dots z'_{k'}, u'_1 \dots u'_{\ell'}) \text{ in } V' \xrightarrow{T'} (0, \dots, 0, w'_1 = T'u'_1, \dots, w'_\ell = T'u'_\ell, y'_1 \dots y'_{n'}) \text{ in } W'$$

by the given matching of ranks & dimensions, $k = k'$, $\ell = \ell'$, $n = n'$, so it makes sense to define Φ and Ψ by

$$\Phi : (z_1 \dots z_k, u_1, \dots u_\ell) \longmapsto (z'_1, \dots, z'_k, u'_1, \dots, u'_\ell)$$

$$\Psi : (w_1, \dots w_\ell, y_1, \dots, y_n) \longmapsto (w'_1, \dots, w'_\ell, y'_1, \dots, y'_n)$$

Now note that Φ and Ψ are isomorphisms and that

$\psi \circ T = T' \circ \Phi$, because this holds on a basis, and if two l.t. are equal on a basis, they are equal everywhere.

on board

Something about all l.t:

Reminder: Choosing a basis, V is isomorphic to F^n.

Goal: 1. The set $\mathcal{L}(V,W)$ of all lin. trans. $V \to W$ is a vector space.

2. Choosing bases, it is isomorphic to $M_{m \times n}$ $(m = \dim W,\ n = \dim V)$

Then follow October 26, 2006:

Let $\beta = (u_1 \ldots u_n)$ be an <u>ordered</u> basis of a f.d. V.S. V. If $x = \sum a_i u_i$, write

$$[x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \left(= Tx, \text{ if } T \text{ is the iso. } V \to F^n \text{ given by } u_i \to e_i \right)$$

The coords of x rel. to β.

<u>Example</u> in $P_2(\mathbb{R})$ $\quad [x^2-2x+3]_{(1,x,x^2)} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

<u>Def</u> Given $T: V \to W$ a lin trans, and ordered basis $\beta = (v_1 \ldots v_n)$ of V & $\gamma = (w_1, \ldots w_m)$ of W,

Let

$$A = [T]_\beta^\gamma = \left([Tv_1]_\gamma \middle| [Tv_2]_\gamma \middle| \ldots \middle| [Tv_n]_\gamma \right) \in M_{m \times n}(F)$$

<u>Note</u> 1. T can be reconstructed from $[T]_\beta^\gamma$

2. Every matrix arises in this way.

$\quad 0:0 \quad 1:1$

<u>Examples</u> 2. $D: P_3(\mathbb{R}) \to P_2(\mathbb{R})$ by differentiation.

3 T_θ

<u>Def</u> $\mathcal{L}(V, W)$

<u>Claim</u> 1. $\mathcal{L}(V, W)$ is a V.S.

2. $T \longmapsto [T]_\beta^\gamma$ is an isomorphism of V.S.,

$$\mathcal{L}(V, W) \longrightarrow M_{m \times n}(F)$$