Lemma (the replacement lemma)

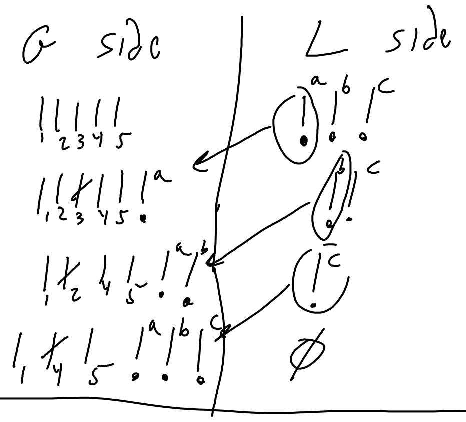
$|G|=n$, $\operatorname{span} G = V$, L lin indep

$\Rightarrow |L| \le n$ & $\exists R \subset G$ with

$|R| = |L|$ and $\operatorname{span}((G \setminus R) \cup L) = V$

Informal proof First of all, if $\sum a_i u_i = 0$, the any vector that appears in this dependency with non-zero coeff is a l.c. of the others.

Now:



Formal proof: Induction on $|L|$. $|L|=0$: trivial.

Now $|L|=m+1$; $L=\{v_1 \dots v_{m+1}\}$. Use $L'=\{v_1 \dots v_m\}$, Find $H'=\{u_1 \dots u_{n-m}\} \subset G$ s.t. $\{u_1 \dots u_{n-m}, v_1 \dots v_m\}$ spans. write

$$v_{m+1} = a_1 u_1 + \dots + a_{n-m} u_{n-m} + b_1 v_1 + \dots + b_m v_m$$

$\therefore$ Not all $a_i = 0$, so $n-m > 0$, so $m+1 \le n$.

$\therefore$ w.l.o.g. $a_1 \ne 0$, so $u_1 \in \operatorname{span}(u_2, \dots u_{n-m}, v_1 \dots v_{m+1})$,

So take $H = \{u_2 \dots u_{n-m}\}$.

Corollaries: 1. If V has a finite basis β_1 then every other basis β_2 of V is also finite & $|\beta_1| = |\beta_2|$.

2. "$\dim V$" makes sense.

hour 13

3. Assume $\dim V = n$. Then   hour 14

a. If G generates V, $|G| \geq n$ & if also $|G| = n$, then G is a basis.

b. If L is linearly indep in V, the $|L| \leq n$;
if also $|L| = n$, L is a basis.
if also $|L| < n$, L can be extended to a basis.

4. If V is finite-dimensional and $W \subset V$ is a subspace, then W is f.d. and $\dim W \leq \dim V$.
If also $\dim W = \dim V$, then $W = V$.
If also $\dim W < \dim V$, then any basis of W can be extended to a basis of V.