AKT-091020, Hour 16: gr, fil, expansions

October-19-09
6:30 PM

1. HW2 2. Next class fun.

Def A UFTI $Z: K \to A$.

Claim Every $W \in A^*$ comes from a V iff
$\exists$ a UFTI $Z: K \to A$

Filtered v.s.; the category of filtered v.s.,
graded v.s.; the category of graded v.s.,
The Functor $gr: f\text{-Vect} \to g\text{-Vect}$.
The Functor $fil: g\text{-Vect} \to f\text{-Vect}$
$gr \circ fil \cong Id$ but $fil \circ gr \not\cong Id$

"Adjoint"?: $K \in f\text{-vect}$, $A \in g\text{-vect}$

$Hom_{g\text{-vect}}(grK, A) \overset{?}{\cong} Hom_{f\text{-vect}}(K, fil A)$

$\leftarrow$ This map is obvious
$\rightarrow$: ?

$Hom_g(A, grK) \overset{?}{\cong} Hom_f(fil A, K)$
$\leftarrow$ obvious.
$\rightarrow$?

Expansion: for $K \in f\text{-Vect}$ $Z: K \to fil(gr(K))$ s.t.
$gr Z: grK \to grK$ is I.

done line

Claim An expansion always exists. [in vect]

Pf

$$K_0 \supset K_1 \supset K_2 \supset K_3$$
$$K_0/K_1 \oplus K_1/K_2 \oplus K_2/K_3 \supset K_3/K_4$$

Find $K_1^\perp$ in K_0
$K_n^\perp$ in K_{n-1}
so $K_{n-1}/K_n \cong K_n^\perp$

$K_0 \cong K_1^\perp \oplus K_1 \cong \dots$
$\cong K_1^\perp \oplus K_2^\perp \oplus K_3^\perp \oplus \dots \oplus K_n$
$\quad\; \downarrow \qquad \downarrow \qquad \downarrow$
$\quad K_0/K_1 \quad K_1/K_2 \quad K_3/K_3$

An A-expansion for $K \in f\text{-vect}$, given $A \in g\text{-vect}$ and a surjective $\pi: A \to grK$, is
$Z: K \to fil A$ s.t.

$$\begin{tikzcd} gr K \arrow[r, "gr Z"] \arrow[dr, "I"'] & A \arrow[d, "\pi"] \\ & gr K \end{tikzcd}$$

An A-expansion does two things:

1. Proves that π is an iso.
2. Constructs an expansion.

If time, guess maps $m: K \otimes K \to K$, $\Delta: K(\uparrow) \to K(\uparrow\uparrow)$ and $\square: K(\uparrow) \to K(\uparrow) \otimes K(\uparrow)$ s.t. the following diagrams commute:

$$\begin{tikzcd} A(\uparrow) \arrow[r, "\Delta"] \arrow[d, "\pi"] & A(\uparrow\uparrow) \arrow[d, "\pi"] \\ gr K(\uparrow) \arrow[r, "gr\Delta"] & gr K(\uparrow\uparrow) \end{tikzcd}$$

$$\begin{tikzcd} A(\uparrow) \otimes A(\uparrow) \arrow[r, "m"] \arrow[d, "\pi \otimes \pi"] & A(\uparrow) \arrow[d, "\pi"] \\ gr(K) \otimes gr(K) & \\ gr(K \otimes K) \arrow[r] & gr(K) \end{tikzcd}$$

$$\mathrm{gr}\,\check{K}(\uparrow) \xrightarrow{\mathrm{gr}\,\Delta} \mathrm{gr}\,\check{K}(\uparrow\uparrow)$$

$$\begin{tikzcd} \mathrm{gr}(K)\otimes\mathrm{gr}(K) \arrow[d, phantom, "\shortparallel"] & \\ \mathrm{gr}(K\otimes K) \arrow[r, "\mathrm{gr}(m)"'] & \check{\mathrm{gr}}(K) \end{tikzcd}$$

$$\begin{tikzcd} A(\uparrow) \arrow[r, "\square"] \arrow[d, "\pi"] & A(\uparrow)\otimes A(\uparrow) \arrow[d] \\ K(\uparrow) \arrow[dr, "\mathrm{gr}\,\square"] & \mathrm{gr}(K)\otimes\mathrm{gr}(K) \arrow[d, phantom, "\shortparallel"] \\ & \mathrm{gr}(K\otimes K) \end{tikzcd}$$