

The Multiple Zeta Function

```

DefineNMZ := (
  NMZ[s_] := NMZ[s] = NSum[1/n^s, {n, 1, Infinity}];
  NMZ[s1_, s2_] := NMZ[s1, s2] =
    Re[NSum[1/(n1^s1 * n2^s2), {n1, 1, Infinity}, {n2, n1+1, Infinity}]];
  NMZ[ss__] := NMZ[ss] = Module[
    {n, i},
    n[0] = 0; n[i_] := n[i] = Unique[n];
    Re[NSum @@ Join[
      {1/Product[n[i]^ss[[i]], {i, Length[ss]}]},
      Table[
        {n[i], n[i-1]+1, Infinity},
        {i, Length[ss]}
      ]
    ]
  ];
);
DefineNMZ;
MZ /: N[MZ[ss__]] := NMZ[ss]

Sqrt[6 * MZ[2]] // N
3.14159

{MZ[3], Zeta[3]} // N
{1.20206, 1.20206}

{MZ[5] * MZ[7], MZ[5, 7] + MZ[7, 5] + MZ[12]} // N
{1.04559, 1.04559}

{MZ[2] * MZ[4, 8], MZ[2, 4, 8] + MZ[6, 8] + MZ[4, 2, 8] + MZ[4, 10] + MZ[4, 8, 2]} // N
{0.00672498, 0.00672498}

Timing[MZ[2, 2, 2] // N]
{3.062 Second, 0.190743}

```

Timing[MZ[2, 2, 2, 2] // N]

```
SequenceLimit::seqlim: The general form of the
sequence could not be determined, and the result may be incorrect. More...

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General::stop:
Further output of SequenceLimit::seqlim will be suppressed during this calculation. More...

NSum::nsum:
Summand (or its derivative) NSum[<<1>>] is not numerical at point n$95459$95460 = 12. More...

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Summand (or its derivative) NSum[<<1>>] is not numerical at point n$95459$95460 = 12. More...

General::stop: Further output of NSum::nsum will be suppressed during this calculation. More...

{1003.36 Second,
Re[NSum[1 / (n$95459$95460^2 n$95459$95461^2 n$95459$95462^2 n$95459$95463^2),
{n$95459$95460, 1., ∞}, {n$95459$95461, 1.+n$95459$95460, ∞},
{n$95459$95462, 1.+n$95459$95461, ∞}, {n$95459$95463, 1.+n$95459$95462, ∞}]]}]
```

From KZ Integrals to Multiple Zeta Values

```
KZI[] = 1;
KZI2MZV[expr_] := Module[
  {slider, t1, t2},
  t1 = expr /. KZI[cd__] => slider[{}, {cd}];
  t2 = t1 //. {
    slider[{{ss__}, {b, cd__}] => -slider[{{ss, 1}, {cd}},
    slider[{{ss__, n_}, {a, cd__}] => slider[{{ss, n+1}, {cd}}
  ];
  t2 /. slider[{{ss__}, {}] => MZ[ss]
]

KZI[b, a, a, b, b, a] // KZI2MZV
-MZ[3, 1, 2]
```

```

Shuffles[l1_List, {}] = {l1};
Shuffles[{}, l2_List] = {l2};
Shuffles[{x_, l1_}, {y_, l2_}] := Join[
  Prepend[#, x] & /@ Shuffles[{l1}, {y, l2}],
  Prepend[#, y] & /@ Shuffles[{x, l1}, {l2}]
];
Shuffle[KZI[l1_], KZI[l2_]] := Plus @@ (KZI @@@ Shuffles[{l1}, {l2}]);
General::spell1: Possible spelling error: new
symbol name "Shuffle" is similar to existing symbol "Shuffles". More...

Shuffles[{1, 2, 3}, {4, 5}]
{{1, 2, 3, 4, 5}, {1, 2, 4, 3, 5}, {1, 2, 4, 5, 3}, {1, 4, 2, 3, 5}, {1, 4, 2, 5, 3},
{1, 4, 5, 2, 3}, {4, 1, 2, 3, 5}, {4, 1, 2, 5, 3}, {4, 1, 5, 2, 3}, {4, 5, 1, 2, 3}}

t1 = {KZI[b, a] * KZI[b, a, a], Shuffle[KZI[b, a], KZI[b, a, a]]}
{KZI[b, a] KZI[b, a, a], KZI[b, a, a, b, a] + 3 KZI[b, a, b, a, a] + 6 KZI[b, b, a, a, a]}

t2 = t1 // KZI2MZV
{MZ[2] MZ[3], 6 MZ[1, 4] + 3 MZ[2, 3] + MZ[3, 2]}

t2 // N
{1.9773, 1.9773}

Shuffle[0, _] = 0;
Shuffle[_, 0] = 0;
Shuffle[a_Plus, b_] := Shuffle[#, b] & /@ a;
Shuffle[a_, b_Plus] := Shuffle[a, #] & /@ b;
Shuffle[a1_ * kzi1_KZI, a2_ * kzi2_KZI] := Expand[a1 a2 Shuffle[kzi1, kzi2]];
KZIReduce[expr_] := FixedPoint[
  (Expand /@ (# // {
    kzi1_KZI * kzi2_KZI => Shuffle[kzi1, kzi2],
    kzi_KZI^p_Integer => Nest[Shuffle[#, kzi] &, kzi, p - 1]
  }))] &,
  expr
];
KZIReduce[KZI[b, a] * KZI[b, a, a]]
KZI[b, a, a, b, a] + 3 KZI[b, a, b, a, a] + 6 KZI[b, b, a, a, a]

{t1 = KZI[b, b, a] + KZI[b, a, a] // KZI2MZV, t1 // N}
{-MZ[3] + MZ[1, 2], -0.00294651}

SetOptions[NSum, NSumTerms -> 50, NSumExtraTerms -> 25]; Clear[NMZ]; DefineNMZ;
{t1 = KZI[b, b, a] + KZI[b, a, a] // KZI2MZV, t1 // N}
{-MZ[3] + MZ[1, 2], -0.000989771}

```

```

SetOptions[NSum, NSumTerms -> 15, NSumExtraTerms -> 12]; Clear[NMZ]; DefineNMZ;

{t1 = KZI[b, b, b, a] - KZI[b, a, a, a] // KZI2MZV, t1 // N}

SequenceLimit::seqlim: The general form of the
sequence could not be determined, and the result may be incorrect. More...

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sequence could not be determined, and the result may be incorrect. More...

{MZ[4] - MZ[1, 1, 2], 0.0286096}

t1 = KZI[b, a, a, b, a, a, a, a] - KZI[b, b, b, b, a, b, b, a] // KZI2MZV
MZ[3, 5] - MZ[1, 1, 1, 2, 1, 2]

t1 // N
$Aborted

```

The Naive KZ Associator

```

Words[alphabeth_List, 0] = {Word[]};
Words[alphabeth_List, n_] :=
  Flatten[Outer[Word, Sequence @@ Table[alphabeth, {n}]]]

General::spell1: Possible spelling error: new symbol name "Words" is similar to existing symbol "Word". More...

Words[{a, b}, 3]

{Word[a, a, a], Word[a, a, b], Word[a, b, a], Word[a, b, b],
 Word[b, a, a], Word[b, a, b], Word[b, b, a], Word[b, b, b]}

PhiNaive[n_] := ASeries @@ Table[
  Plus @@ (Words[{a, b}, k] /. Word[w___] -> KZI[w] * P[w]),
  {k, 0, n}
]

General::spell1: Possible spelling error: new
symbol name "ASeries" is similar to existing symbol "Series". More...

PhiNaive[2]

ASeries[P[], KZI[a] P[a] + KZI[b] P[b],
 KZI[a, a] P[a, a] + KZI[a, b] P[a, b] + KZI[b, a] P[b, a] + KZI[b, b] P[b, b]]

```

Some Non-Commutative Algebra

```

Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[] := P[];
NonCommutativeMultiply[a_] := a;
0 ** _ = 0;
_ ** 0 = 0;
(a_Plus) ** b_ := (#**b) & /@ a;
a_ ** (b_Plus) := (a**#) & /@ b;
(a1_. * p1_P) ** (a2_. * p2_P) := Expand[a1 a2 Join[p1, p2]];

ASeries /: a_ * ASeries[ps__] := ASeries[Sequence @@ (a * {ps})];
ASeries /: ASeries[ps1__] + ASeries[ps2__] := Module[
  {d},
  d = Min[Length[{ps1}], Length[{ps2}]];
  ASeries[Sequence @@ (Take[{ps1}, d] + Take[{ps2}, d])]
];
ASeries /: ASeries[ps1__] ** ASeries[ps2__] := Module[
  {d, k, i},
  d = Min[Length[{ps1}], Length[{ps2}]] - 1;
  ASeries[Sequence @@ Table[
    Sum[
      {ps1}[[1+i]] ** {ps2}[[1+k-i]],
      {i, 0, k}
    ],
    {k, 0, d}
  ]]
];
ASeries /: Expand[as_ASeries] := Expand /@ as

PhiNaive[2] + PhiNaive[3]
ASeries[2 P[], 2 KZI[a] P[a] + 2 KZI[b] P[b],
  2 KZI[a, a] P[a, a] + 2 KZI[a, b] P[a, b] + 2 KZI[b, a] P[b, a] + 2 KZI[b, b] P[b, b]]

PhiNaive[2] ** PhiNaive[3]
ASeries[P[], 2 KZI[a] P[a] + 2 KZI[b] P[b],
  KZI[a]^2 P[a, a] + 2 KZI[a, a] P[a, a] + KZI[a] KZI[b] P[a, b] + 2 KZI[a, b] P[a, b] +
  KZI[a] KZI[b] P[b, a] + 2 KZI[b, a] P[b, a] + KZI[b]^2 P[b, b] + 2 KZI[b, b] P[b, b]]

```

The Renormalized KZ Associator

```
Phi[n_] := KZIReduce[
  NonCommutativeMultiply[
    ASeries @@ Table[
      (-1)^k * (KZI @@ Table[a, {k}]) (P @@ Table[a, {k}]),
      {k, 0, n}
    ],
    PhiNaive[n],
    ASeries @@ Table[
      (-1)^k * (KZI @@ Table[b, {k}]) (P @@ Table[b, {k}]),
      {k, 0, n}
    ]
  ]
]
```

Phi[2]

```
ASeries[P[], 0, -KZI[b, a] P[a, b] + KZI[b, a] P[b, a]]
```

Phi[4]

```
ASeries[P[], 0, -KZI[b, a] P[a, b] + KZI[b, a] P[b, a],
  KZI[b, a, a] P[a, a, b] - 2 KZI[b, a, a] P[a, b, a] + KZI[b, b, a] P[a, b, b] +
  KZI[b, a, a] P[b, a, a] - 2 KZI[b, b, a] P[b, a, b] + KZI[b, b, a] P[b, b, a],
  -KZI[b, a, a, a] P[a, a, a, b] + 3 KZI[b, a, a, a] P[a, a, b, a] -
  KZI[b, b, a, a] P[a, a, b, b] - 3 KZI[b, a, a, a] P[a, b, a, a] +
  KZI[b, a, b, a] P[a, b, a, b] + 4 KZI[b, b, a, a] P[a, b, a, b] -
  KZI[b, a, b, a] P[a, b, b, a] - 2 KZI[b, b, a, a] P[a, b, b, a] -
  KZI[b, b, b, a] P[a, b, b, b] + KZI[b, a, a, a] P[b, a, a, a] -
  KZI[b, a, b, a] P[b, a, a, b] - 2 KZI[b, b, a, a] P[b, a, a, b] +
  KZI[b, a, b, a] P[b, a, b, a] + 3 KZI[b, b, b, a] P[b, a, b, b] + KZI[b, b, a, a]
  P[b, b, a, a] - 3 KZI[b, b, b, a] P[b, b, a, b] + KZI[b, b, b, a] P[b, b, b, a]]
```

Phi[5] // KZI2MZV

```

ASeries[P[], 0, MZ[2] P[a, b] - MZ[2] P[b, a],
  -MZ[3] P[a, a, b] + 2 MZ[3] P[a, b, a] + MZ[1, 2] P[a, b, b] - MZ[3] P[b, a, a] -
    2 MZ[1, 2] P[b, a, b] + MZ[1, 2] P[b, b, a], MZ[4] P[a, a, a, b] - 3 MZ[4] P[a, a, b, a] -
    MZ[1, 3] P[a, a, b, b] + 3 MZ[4] P[a, b, a, a] + 4 MZ[1, 3] P[a, b, a, b] +
    MZ[2, 2] P[a, b, a, b] - 2 MZ[1, 3] P[a, b, b, a] - MZ[2, 2] P[a, b, b, a] +
    MZ[1, 1, 2] P[a, b, b, b] - MZ[4] P[b, a, a, a] - 2 MZ[1, 3] P[b, a, a, b] -
    MZ[2, 2] P[b, a, a, b] + MZ[2, 2] P[b, a, b, a] - 3 MZ[1, 1, 2] P[b, a, b, b] +
    MZ[1, 3] P[b, b, a, a] + 3 MZ[1, 1, 2] P[b, b, a, b] - MZ[1, 1, 2] P[b, b, b, a],
  -MZ[5] P[a, a, a, a, b] + 4 MZ[5] P[a, a, a, b, a] + MZ[1, 4] P[a, a, a, b, b] -
    6 MZ[5] P[a, a, b, a, a] - 6 MZ[1, 4] P[a, a, b, a, b] - 2 MZ[2, 3] P[a, a, b, a, b] -
    MZ[3, 2] P[a, a, b, a, b] + 3 MZ[1, 4] P[a, a, b, b, a] + 2 MZ[2, 3] P[a, a, b, b, a] +
    MZ[3, 2] P[a, a, b, b, a] - MZ[1, 1, 3] P[a, a, b, b, b] + 4 MZ[5] P[a, b, a, a, a] +
    6 MZ[1, 4] P[a, b, a, a, b] + 3 MZ[2, 3] P[a, b, a, a, b] + 2 MZ[3, 2] P[a, b, a, a, b] -
    2 MZ[2, 3] P[a, b, a, b, a] - 2 MZ[3, 2] P[a, b, a, b, a] + 6 MZ[1, 1, 3] P[a, b, a, b, b] +
    2 MZ[1, 2, 2] P[a, b, a, b, b] + MZ[2, 1, 2] P[a, b, a, b, b] -
    3 MZ[1, 4] P[a, b, b, a, a] - MZ[2, 3] P[a, b, b, a, a] - 6 MZ[1, 1, 3] P[a, b, b, a, b] -
    3 MZ[1, 2, 2] P[a, b, b, a, b] - 2 MZ[2, 1, 2] P[a, b, b, a, b] +
    2 MZ[1, 1, 3] P[a, b, b, b, a] + MZ[1, 2, 2] P[a, b, b, b, a] +
    MZ[2, 1, 2] P[a, b, b, b, a] + MZ[1, 1, 1, 2] P[a, b, b, b, b] -
    MZ[5] P[b, a, a, a, a] - 2 MZ[1, 4] P[b, a, a, a, b] - MZ[2, 3] P[b, a, a, a, b] -
    MZ[3, 2] P[b, a, a, a, b] + MZ[3, 2] P[b, a, a, b, a] - 3 MZ[1, 1, 3] P[b, a, a, b, b] -
    2 MZ[1, 2, 2] P[b, a, a, b, b] - MZ[2, 1, 2] P[b, a, a, b, b] + MZ[2, 3] P[b, a, b, a, a] +
    2 MZ[1, 2, 2] P[b, a, b, a, b] + 2 MZ[2, 1, 2] P[b, a, b, a, b] -
    MZ[2, 1, 2] P[b, a, b, b, a] - 4 MZ[1, 1, 1, 2] P[b, a, b, b, b] + MZ[1, 4] P[b, b, a, a, a] +
    3 MZ[1, 1, 3] P[b, b, a, a, b] + MZ[1, 2, 2] P[b, b, a, a, b] - MZ[1, 2, 2] P[b, b, a, b, a] +
    6 MZ[1, 1, 1, 2] P[b, b, a, b, b] - MZ[1, 1, 3] P[b, b, b, a, a] -
    4 MZ[1, 1, 1, 2] P[b, b, b, a, b] + MZ[1, 1, 1, 2] P[b, b, b, b, a]]

```

Phi[3] // KZI2MZV // N

```

ASeries[P[], 0., 1.64493 P[a, b] - 1.64493 P[b, a],
  -1.20206 P[a, a, b] + 2.40411 P[a, b, a] + 1.19911 P[a, b, b] -
    1.20206 P[b, a, a] - 2.39822 P[b, a, b] + 1.19911 P[b, b, a]]

```

The VS Structure and the Pentagon

```

Pull[l_List, t[i_, j_]] := Plus @@ (Plus @@ (
  Outer[t, Flatten[Position[l, i]], Flatten[Position[l, j]]] /.
    t[i1_, j1_] /; i1 > j1 => t[j1, i1]
));
Pull[l_List, a] := Pull[l, t[1, 2]];
Pull[l_List, b] := Pull[l, t[2, 3]];
Pull[l_List, expr_] := Expand[expr /.
  p_P => NonCommutativeMultiply @@ ((Pull[l, #] & /@ p) /. t[ij_] => P[t[ij]])]

```

General::spell: Possible spelling error: new
symbol name "Pull" is similar to existing symbols {Full, Null}. More...

```
Pull[{1, 2, 2, 3}, b]
```

```
t[2, 4] + t[3, 4]
```

```
Pull[{1, 2, 2, 3}, Phi[2]] // Expand
```

```

ASeries[P[], 0, -KZI[b, a] P[t[1, 2], t[2, 4]] - KZI[b, a] P[t[1, 2], t[3, 4]] -
  KZI[b, a] P[t[1, 3], t[2, 4]] - KZI[b, a] P[t[1, 3], t[3, 4]] +
  KZI[b, a] P[t[2, 4], t[1, 2]] + KZI[b, a] P[t[2, 4], t[1, 3]] +
  KZI[b, a] P[t[3, 4], t[1, 2]] + KZI[b, a] P[t[3, 4], t[1, 3]]]

```

```
Pull[{0, 0, 2, 3}, PhiNaive[3]] // Expand
```

```

ASeries[P[], KZI[b] P[t[3, 4]],
  KZI[b, b] P[t[3, 4], t[3, 4]], KZI[b, b, b] P[t[3, 4], t[3, 4], t[3, 4]]]

```

```

TestPentagon[Phi_] := (
  Pull[{1, 2, 3, 0}, Phi] ** Pull[{1, 2, 2, 3}, Phi] ** Pull[{0, 1, 2, 3}, Phi] -
  Pull[{1, 1, 2, 3}, Phi] ** Pull[{1, 2, 3, 3}, Phi]
)

```

```
TestPentagon[Phi[2]]
```

```

ASeries[0, 0, -KZI[b, a] P[t[1, 2], t[3, 4]] - KZI[b, a] P[t[1, 3], t[2, 4]] +
  KZI[b, a] P[t[2, 4], t[1, 3]] + KZI[b, a] P[t[3, 4], t[1, 2]]]

```


The 4T Relations

```

Normalize[expr_] := Expand[expr //. {
  P[lft___, t[i_, j_], t[k_, l_], rgt___] => Which[
    i == k, P[lft, t[k, l], t[i, j], rgt] +
      P[lft, t[l, j], t[i, j], rgt] - P[lft, t[i, j], t[l, j], rgt],
    i == l, P[lft, t[k, l], t[i, j], rgt] + P[lft, t[k, j], t[i, j], rgt] -
      P[lft, t[i, j], t[k, j], rgt],
    True, P[lft, t[k, l], t[i, j], rgt]
  ] /; j > l
}]

TestPentagon[Phi[2]] // Normalize
ASeries[0, 0, 0]

t1 = TestPentagon[Phi[3]] // Normalize
ASeries[0, 0, 0, -2 KZI[b, a, a] P[t[1, 4], t[2, 4], t[3, 4]] -
  2 KZI[b, b, a] P[t[1, 4], t[2, 4], t[3, 4]] +
  KZI[b, a, a] P[t[1, 4], t[3, 4], t[2, 4]] + KZI[b, b, a] P[t[1, 4], t[3, 4], t[2, 4]] +
  KZI[b, a, a] P[t[2, 4], t[1, 4], t[3, 4]] + KZI[b, b, a] P[t[2, 4], t[1, 4], t[3, 4]] +
  KZI[b, a, a] P[t[2, 4], t[3, 4], t[1, 4]] + KZI[b, b, a] P[t[2, 4], t[3, 4], t[1, 4]] +
  KZI[b, a, a] P[t[3, 4], t[1, 4], t[2, 4]] + KZI[b, b, a] P[t[3, 4], t[1, 4], t[2, 4]] -
  2 KZI[b, a, a] P[t[3, 4], t[2, 4], t[1, 4]] - 2 KZI[b, b, a] P[t[3, 4], t[2, 4], t[1, 4]]]

t2 = t1 // KZI2MZV
ASeries[0, 0, 0,
  2 MZ[3] P[t[1, 4], t[2, 4], t[3, 4]] - 2 MZ[1, 2] P[t[1, 4], t[2, 4], t[3, 4]] -
  MZ[3] P[t[1, 4], t[3, 4], t[2, 4]] + MZ[1, 2] P[t[1, 4], t[3, 4], t[2, 4]] -
  MZ[3] P[t[2, 4], t[1, 4], t[3, 4]] + MZ[1, 2] P[t[2, 4], t[1, 4], t[3, 4]] -
  MZ[3] P[t[2, 4], t[3, 4], t[1, 4]] + MZ[1, 2] P[t[2, 4], t[3, 4], t[1, 4]] -
  MZ[3] P[t[3, 4], t[1, 4], t[2, 4]] + MZ[1, 2] P[t[3, 4], t[1, 4], t[2, 4]] +
  2 MZ[3] P[t[3, 4], t[2, 4], t[1, 4]] - 2 MZ[1, 2] P[t[3, 4], t[2, 4], t[1, 4]]]

t2 /. mz_MZ => N[mz]
ASeries[0, 0, 0,
  0.00589302 P[t[1, 4], t[2, 4], t[3, 4]] - 0.00294651 P[t[1, 4], t[3, 4], t[2, 4]] -
  0.00294651 P[t[2, 4], t[1, 4], t[3, 4]] - 0.00294651 P[t[2, 4], t[3, 4], t[1, 4]] -
  0.00294651 P[t[3, 4], t[1, 4], t[2, 4]] + 0.00589302 P[t[3, 4], t[2, 4], t[1, 4]]]

Coefficient[Last[t2], #] & /@ Union[Cases[Last[t2], _P, Infinity]]
{2 MZ[3] - 2 MZ[1, 2], -MZ[3] + MZ[1, 2], -MZ[3] + MZ[1, 2],
  -MZ[3] + MZ[1, 2], -MZ[3] + MZ[1, 2], 2 MZ[3] - 2 MZ[1, 2]}

```

```

t1 = TestPentagon[Phi[4]] // Normalize;
t2 = t1 // KZI2MZV;
t2 /. mz_MZ -> N[mz]
ASeries[0, 0, 0,
0.00589302 P[t[1, 4], t[2, 4], t[3, 4]] - 0.00294651 P[t[1, 4], t[3, 4], t[2, 4]] -
0.00294651 P[t[2, 4], t[1, 4], t[3, 4]] - 0.00294651 P[t[2, 4], t[3, 4], t[1, 4]] -
0.00294651 P[t[3, 4], t[1, 4], t[2, 4]] + 0.00589302 P[t[3, 4], t[2, 4], t[1, 4]],
0.00015359 P[t[1, 3], t[2, 3], t[1, 4], t[2, 4]] -
0.00015359 P[t[1, 3], t[2, 3], t[2, 4], t[1, 4]] +
4.27574 x 10-6 P[t[1, 4], t[1, 4], t[2, 4], t[3, 4]] -
0.0000746591 P[t[1, 4], t[1, 4], t[3, 4], t[2, 4]] -
4.27574 x 10-6 P[t[1, 4], t[2, 4], t[1, 4], t[3, 4]] +
0.0858246 P[t[1, 4], t[2, 4], t[2, 4], t[3, 4]] -
4.27574 x 10-6 P[t[1, 4], t[2, 4], t[3, 4], t[1, 4]] -
0.0859035 P[t[1, 4], t[2, 4], t[3, 4], t[2, 4]] -
4.27574 x 10-6 P[t[1, 4], t[2, 4], t[3, 4], t[3, 4]] -
8.54781 x 10-6 P[t[1, 4], t[3, 4], t[1, 4], t[2, 4]] +
0.000157866 P[t[1, 4], t[3, 4], t[2, 4], t[1, 4]] +
0.0286843 P[t[1, 4], t[3, 4], t[2, 4], t[2, 4]] +
0.114517 P[t[1, 4], t[3, 4], t[2, 4], t[3, 4]] -
0.0572982 P[t[1, 4], t[3, 4], t[3, 4], t[2, 4]] -
0.00015359 P[t[2, 3], t[1, 3], t[1, 4], t[2, 4]] +
0.00015359 P[t[2, 3], t[1, 3], t[2, 4], t[1, 4]] +
0.0000746591 P[t[2, 4], t[1, 4], t[1, 4], t[3, 4]] -
0.0857457 P[t[2, 4], t[1, 4], t[2, 4], t[3, 4]] -
0.000145042 P[t[2, 4], t[1, 4], t[3, 4], t[1, 4]] +
0.028535 P[t[2, 4], t[1, 4], t[3, 4], t[2, 4]] -
0.057215 P[t[2, 4], t[1, 4], t[3, 4], t[3, 4]] +
0.0286054 P[t[2, 4], t[2, 4], t[1, 4], t[3, 4]] -
0.0286843 P[t[2, 4], t[2, 4], t[3, 4], t[1, 4]] +
0.0000746591 P[t[2, 4], t[3, 4], t[1, 4], t[1, 4]] -
0.028535 P[t[2, 4], t[3, 4], t[1, 4], t[2, 4]] +
0.0000746591 P[t[2, 4], t[3, 4], t[1, 4], t[3, 4]] +
0.0859035 P[t[2, 4], t[3, 4], t[2, 4], t[1, 4]] +
0.0571446 P[t[2, 4], t[3, 4], t[3, 4], t[1, 4]] +
0.0000789311 P[t[3, 4], t[1, 4], t[1, 4], t[2, 4]] -
0.000149314 P[t[3, 4], t[1, 4], t[2, 4], t[1, 4]] -
0.0286054 P[t[3, 4], t[1, 4], t[2, 4], t[2, 4]] -
0.114509 P[t[3, 4], t[1, 4], t[2, 4], t[3, 4]] +
0.0000789311 P[t[3, 4], t[1, 4], t[3, 4], t[2, 4]] -
4.27574 x 10-6 P[t[3, 4], t[2, 4], t[1, 4], t[1, 4]] +
0.0857457 P[t[3, 4], t[2, 4], t[1, 4], t[2, 4]] +

```

```

0.114355 P[t[3, 4], t[2, 4], t[1, 4], t[3, 4]] -
0.0858246 P[t[3, 4], t[2, 4], t[2, 4], t[1, 4]] -
0.114364 P[t[3, 4], t[2, 4], t[3, 4], t[1, 4]] +
0.057215 P[t[3, 4], t[3, 4], t[1, 4], t[2, 4]] +
4.27574 × 10-6 P[t[3, 4], t[3, 4], t[2, 4], t[1, 4]]]

```

```
Union[Coefficient[Last[t2], #] & /@ Union[Cases[Last[t2], _P, Infinity]]]
```

```

{-MZ[2]^2 + 3 MZ[4] - 2 MZ[1, 3], MZ[2]^2 - 3 MZ[4] + 2 MZ[1, 3], -MZ[4] + 4 MZ[1, 3],
3 MZ[4] - 6 MZ[1, 3] - 2 MZ[2, 2], MZ[2]^2 - 4 MZ[1, 3] - 2 MZ[2, 2],
MZ[2]^2 - MZ[4] - 3 MZ[1, 3] - MZ[2, 2], MZ[4] - MZ[1, 3] - MZ[2, 2],
-MZ[4] + MZ[1, 3] + MZ[2, 2], -MZ[2]^2 + MZ[4] + 2 MZ[2, 2],
-2 MZ[2]^2 + 3 MZ[4] + 2 MZ[1, 3] + 2 MZ[2, 2], -MZ[2]^2 + 4 MZ[1, 3] + 2 MZ[2, 2],
MZ[2]^2 + 3 MZ[4] - 3 MZ[1, 3] - MZ[2, 2] - 4 MZ[1, 1, 2],
2 MZ[2]^2 - 3 MZ[4] + 5 MZ[1, 3] + MZ[2, 2] - 4 MZ[1, 1, 2],
MZ[2]^2 + 2 MZ[1, 3] - 3 MZ[1, 1, 2], 4 MZ[4] - MZ[1, 3] - MZ[2, 2] - 3 MZ[1, 1, 2],
2 MZ[2]^2 - 4 MZ[4] + 5 MZ[1, 3] + MZ[2, 2] - 3 MZ[1, 1, 2],
MZ[2]^2 - MZ[4] + 2 MZ[1, 3] - 2 MZ[1, 1, 2], MZ[4] + MZ[1, 3] + MZ[2, 2] - 2 MZ[1, 1, 2],
MZ[2]^2 - 2 MZ[4] + 2 MZ[1, 3] - MZ[1, 1, 2], 2 MZ[4] - MZ[1, 3] - MZ[2, 2] - MZ[1, 1, 2],
MZ[1, 3] + MZ[2, 2] - MZ[1, 1, 2], -MZ[2]^2 + 2 MZ[4] - 2 MZ[1, 3] + MZ[1, 1, 2],
-MZ[1, 3] - MZ[2, 2] + MZ[1, 1, 2], -2 MZ[4] + MZ[1, 3] + MZ[2, 2] + MZ[1, 1, 2],
-MZ[2]^2 + MZ[4] - 2 MZ[1, 3] + 2 MZ[1, 1, 2],
-MZ[2]^2 - MZ[4] + 3 MZ[1, 3] + MZ[2, 2] + 2 MZ[1, 1, 2], -MZ[2]^2 - 2 MZ[1, 3] + 3 MZ[1, 1, 2],
-2 MZ[2]^2 + 4 MZ[4] - 5 MZ[1, 3] - MZ[2, 2] + 3 MZ[1, 1, 2],
-4 MZ[4] + MZ[1, 3] + MZ[2, 2] + 3 MZ[1, 1, 2], -3 MZ[4] - MZ[1, 3] - MZ[2, 2] + 4 MZ[1, 1, 2],
-3 MZ[2]^2 + 3 MZ[4] - MZ[1, 3] + MZ[2, 2] + 4 MZ[1, 1, 2]}

```

```
Coefficient[Last[t2], P[t[3, 4], t[2, 4], t[3, 4], t[1, 4]]]
```

```
-3 MZ[4] - MZ[1, 3] - MZ[2, 2] + 4 MZ[1, 1, 2]
```

```
-3 MZ[4] - MZ[1, 3] - MZ[2, 2] + 4 MZ[1, 1, 2] // N
```

```
-0.114364
```

```
SetOptions[NSum, NSumTerms → 50, NSumExtraTerms → 25]; Clear[NMZ]; DefineNMZ;
```

```
-3 MZ[4] - MZ[1, 3] - MZ[2, 2] + 4 MZ[1, 1, 2] // N
```

```
SequenceLimit::seqlim: The general form of the
sequence could not be determined, and the result may be incorrect. More...
```

```
SequenceLimit::seqlim: The general form of the
sequence could not be determined, and the result may be incorrect. More...
```

```
SequenceLimit::seqlim: The general form of the
sequence could not be determined, and the result may be incorrect. More...
```

```
General::stop:
Further output of SequenceLimit::seqlim will be suppressed during this calculation. More...
```

```
-0.0688928
```

```
SetOptions[NSum, NSumTerms → 15, NSumExtraTerms → 12]; Clear[NMZ]; DefineNMZ;
```

A Rational Associator

```

phi4 = ASeries[
  P[], 0,
  P[t[1, 3], t[2, 3]] / 24 - P[t[2, 3], t[1, 3]] / 24, 0,
  -P[t[1, 3], t[1, 3], t[1, 3], t[2, 3]] / 1440 +
  P[t[1, 3], t[1, 3], t[2, 3], t[1, 3]] / 480 +
  (7 * P[t[1, 3], t[1, 3], t[2, 3], t[2, 3]]) / 5760 -
  P[t[1, 3], t[2, 3], t[1, 3], t[1, 3]] / 480 -
  P[t[1, 3], t[2, 3], t[1, 3], t[2, 3]] / 640 -
  P[t[1, 3], t[2, 3], t[2, 3], t[1, 3]] / 1152 -
  (7 * P[t[1, 3], t[2, 3], t[2, 3], t[2, 3]]) / 5760 +
  P[t[2, 3], t[1, 3], t[1, 3], t[1, 3]] / 1440 -
  P[t[2, 3], t[1, 3], t[1, 3], t[2, 3]] / 1152 +
  (19 * P[t[2, 3], t[1, 3], t[2, 3], t[1, 3]]) / 5760 +
  (7 * P[t[2, 3], t[1, 3], t[2, 3], t[2, 3]]) / 1920 -
  (7 * P[t[2, 3], t[2, 3], t[1, 3], t[1, 3]]) / 5760 -
  (7 * P[t[2, 3], t[2, 3], t[1, 3], t[2, 3]]) / 1920 +
  (7 * P[t[2, 3], t[2, 3], t[2, 3], t[1, 3]]) / 5760
]
ASeries[P[], 0,  $\frac{1}{24}$  P[t[1, 3], t[2, 3]] -  $\frac{1}{24}$  P[t[2, 3], t[1, 3]], 0,
-  $\frac{P[t[1, 3], t[1, 3], t[1, 3], t[2, 3]]}{1440}$  +  $\frac{1}{480}$  P[t[1, 3], t[1, 3], t[2, 3], t[1, 3]] +
 $\frac{7 P[t[1, 3], t[1, 3], t[2, 3], t[2, 3]]}{5760}$  -  $\frac{1}{480}$  P[t[1, 3], t[2, 3], t[1, 3], t[1, 3]] -
 $\frac{1}{640}$  P[t[1, 3], t[2, 3], t[1, 3], t[2, 3]] -  $\frac{P[t[1, 3], t[2, 3], t[2, 3], t[1, 3]]}{1152}$  -
 $\frac{7 P[t[1, 3], t[2, 3], t[2, 3], t[2, 3]]}{5760}$  +  $\frac{P[t[2, 3], t[1, 3], t[1, 3], t[1, 3]]}{1440}$  -
 $\frac{P[t[2, 3], t[1, 3], t[1, 3], t[2, 3]]}{1152}$  +  $\frac{19 P[t[2, 3], t[1, 3], t[2, 3], t[1, 3]]}{5760}$  +
 $\frac{7 P[t[2, 3], t[1, 3], t[2, 3], t[2, 3]]}{1920}$  -  $\frac{7 P[t[2, 3], t[2, 3], t[1, 3], t[1, 3]]}{5760}$  -
 $\frac{7 P[t[2, 3], t[2, 3], t[1, 3], t[2, 3]]}{1920}$  +  $\frac{7 P[t[2, 3], t[2, 3], t[2, 3], t[1, 3]]}{5760}$ ]

```

TestPentagon[phi4] // Normalize

```
ASeries[0, 0, 0, 0, 0]
```

PhiNaive, the Pentagon and the Septagon

```
Invert[ASeries[P[], ht___]] := Module[
  {x, t},
  x = -ASeries[0, ht];
  t = ASeries[P[], Sequence@@(0 * {ht})];
  t + Sum[
    t = t**x,
    {Length[{ht}}]
  ]
]
```

General::spell1:

Possible spelling error: new symbol name "Invert" is similar to existing symbol "Insert". More...

```
Invert[Phi[4]] ** Phi[4]
```

```
ASeries[P[], 0, 0, 0, 0]
```

```
TestPentagon[PhiNaive[2]] // Normalize // KZIReduce
```

```
ASeries[0, KZI[a] P[t[1, 2]] + KZI[b] P[t[3, 4]],
  3 KZI[a, a] P[t[1, 2], t[1, 2]] + 2 KZI[a, a] P[t[1, 2], t[1, 3]] +
  2 KZI[a, a] P[t[1, 2], t[2, 3]] + KZI[a, b] P[t[1, 2], t[2, 3]] +
  KZI[b, a] P[t[1, 2], t[2, 3]] + KZI[a, b] P[t[1, 2], t[2, 4]] +
  KZI[b, a] P[t[1, 2], t[2, 4]] + 3 KZI[a, b] P[t[1, 2], t[3, 4]] +
  3 KZI[b, a] P[t[1, 2], t[3, 4]] + KZI[a, b] P[t[1, 3], t[3, 4]] +
  KZI[b, a] P[t[1, 3], t[3, 4]] + KZI[a, b] P[t[2, 3], t[3, 4]] +
  KZI[b, a] P[t[2, 3], t[3, 4]] + 2 KZI[b, b] P[t[2, 3], t[3, 4]] +
  2 KZI[b, b] P[t[2, 4], t[3, 4]] + 3 KZI[b, b] P[t[3, 4], t[3, 4]]]
```

```
TestPentagon[Phi_] := (
  Pull[{1, 2, 3, 0}, Phi] ** Pull[{1, 2, 2, 3}, Phi] ** Pull[{0, 1, 2, 3}, Phi] -
  Pull[{1, 1, 2, 3}, Phi] ** Pull[{1, 2, 3, 3}, Phi]
)
```

```

TestSeptagon[Phi_] := Module[
  {n = Length[Phi] - 1},
  NonCommutativeMultiply[
    Invert[Pull[{0, 1, 2, 3}, Phi]],
    Invert[Pull[{1, 2, 2, 3}, Phi]],
    Invert[Pull[{1, 2, 3, 0}, Phi]],
    Pull[{1, 1, 2, 3}, Phi],
    Pull[{0, 0, 2, 3}, Phi],
    Pull[{1, 2, 0, 0}, Phi],
    Pull[{1, 2, 3, 3}, Phi]
  ]
]

t1 = TestSeptagon[PhiNaive[3]] // Normalize // KZIReduce
ASeries[P[], 0, 0,
  2 KZI[b, a, a] P[t[1, 4], t[2, 4], t[3, 4]] + 2 KZI[b, b, a] P[t[1, 4], t[2, 4], t[3, 4]] -
  KZI[b, a, a] P[t[1, 4], t[3, 4], t[2, 4]] - KZI[b, b, a] P[t[1, 4], t[3, 4], t[2, 4]] -
  KZI[b, a, a] P[t[2, 4], t[1, 4], t[3, 4]] - KZI[b, b, a] P[t[2, 4], t[1, 4], t[3, 4]] -
  KZI[b, a, a] P[t[2, 4], t[3, 4], t[1, 4]] - KZI[b, b, a] P[t[2, 4], t[3, 4], t[1, 4]] -
  KZI[b, a, a] P[t[3, 4], t[1, 4], t[2, 4]] - KZI[b, b, a] P[t[3, 4], t[1, 4], t[2, 4]] +
  2 KZI[b, a, a] P[t[3, 4], t[2, 4], t[1, 4]] + 2 KZI[b, b, a] P[t[3, 4], t[2, 4], t[1, 4]]]

(t1 // KZI2MZV) /. mz_MZ -> N[mz]
ASeries[P[], 0, 0,
  -0.00589302 P[t[1, 4], t[2, 4], t[3, 4]] + 0.00294651 P[t[1, 4], t[3, 4], t[2, 4]] +
  0.00294651 P[t[2, 4], t[1, 4], t[3, 4]] + 0.00294651 P[t[2, 4], t[3, 4], t[1, 4]] +
  0.00294651 P[t[3, 4], t[1, 4], t[2, 4]] - 0.00589302 P[t[3, 4], t[2, 4], t[1, 4]]]

SetOptions[NSum, NSumTerms -> 15, NSumExtraTerms -> 12]; Clear[NMZ]; DefineNMZ;
Sort[Abs[Cases[(t1 // KZI2MZV) /. mz_MZ -> N[mz], _Real, Infinity]]]
{0.00294651, 0.00294651, 0.00294651, 0.00294651, 0.00589302, 0.00589302}

SetOptions[NSum, NSumTerms -> 500, NSumExtraTerms -> 250]; Clear[NMZ]; DefineNMZ;
Sort[Abs[Cases[(t1 // KZI2MZV) /. mz_MZ -> N[mz], _Real, Infinity]]]
{0.000354498, 0.000354498, 0.000354498, 0.000354498, 0.000708997, 0.000708997}

SetOptions[NSum, NSumTerms -> 5000, NSumExtraTerms -> 2500]; Clear[NMZ]; DefineNMZ;
Sort[Abs[Cases[(t1 // KZI2MZV) /. mz_MZ -> N[mz], _Real, Infinity]]]
{0.0000351583, 0.0000351583, 0.0000351583, 0.0000351583, 0.0000703165, 0.0000703165}

SetOptions[NSum, NSumTerms -> 15, NSumExtraTerms -> 12]; Clear[NMZ]; DefineNMZ;
t2 = TestSeptagon[PhiNaive[4]] // Normalize // KZIReduce
ASeries[P[], 0, 0,
  2 KZI[b, a, a] P[t[1, 4], t[2, 4], t[3, 4]] + 2 KZI[b, b, a] P[t[1, 4], t[2, 4], t[3, 4]] -

```

$$\begin{aligned}
& KZI[b, a, a] P[t[1, 4], t[3, 4], t[2, 4]] - KZI[b, b, a] P[t[1, 4], t[3, 4], t[2, 4]] - \\
& KZI[b, a, a] P[t[2, 4], t[1, 4], t[3, 4]] - KZI[b, b, a] P[t[2, 4], t[1, 4], t[3, 4]] - \\
& KZI[b, a, a] P[t[2, 4], t[3, 4], t[1, 4]] - KZI[b, b, a] P[t[2, 4], t[3, 4], t[1, 4]] - \\
& KZI[b, a, a] P[t[3, 4], t[1, 4], t[2, 4]] - KZI[b, b, a] P[t[3, 4], t[1, 4], t[2, 4]] + \\
& 2 KZI[b, a, a] P[t[3, 4], t[2, 4], t[1, 4]] + 2 KZI[b, b, a] P[t[3, 4], t[2, 4], t[1, 4]], \\
& 3 KZI[b, a, a, a] P[t[1, 4], t[1, 4], t[2, 4], t[3, 4]] + \\
& 2 KZI[b, a, b, a] P[t[1, 4], t[1, 4], t[2, 4], t[3, 4]] + \\
& 6 KZI[b, b, a, a] P[t[1, 4], t[1, 4], t[2, 4], t[3, 4]] - \\
& KZI[b, a, a, a] P[t[1, 4], t[1, 4], t[3, 4], t[2, 4]] - \\
& KZI[b, a, b, a] P[t[1, 4], t[1, 4], t[3, 4], t[2, 4]] - \\
& KZI[b, b, a, a] P[t[1, 4], t[1, 4], t[3, 4], t[2, 4]] - \\
& 3 KZI[b, a, a, a] P[t[1, 4], t[2, 4], t[1, 4], t[3, 4]] - \\
& 2 KZI[b, a, b, a] P[t[1, 4], t[2, 4], t[1, 4], t[3, 4]] - \\
& 6 KZI[b, b, a, a] P[t[1, 4], t[2, 4], t[1, 4], t[3, 4]] + \\
& 2 KZI[b, a, a, b] P[t[1, 4], t[2, 4], t[2, 4], t[3, 4]] - \\
& 2 KZI[b, b, a, a] P[t[1, 4], t[2, 4], t[2, 4], t[3, 4]] + \\
& 2 KZI[b, b, a, b] P[t[1, 4], t[2, 4], t[2, 4], t[3, 4]] + \\
& 3 KZI[b, b, b, a] P[t[1, 4], t[2, 4], t[2, 4], t[3, 4]] - \\
& 3 KZI[b, a, a, a] P[t[1, 4], t[2, 4], t[3, 4], t[1, 4]] - \\
& 2 KZI[b, a, b, a] P[t[1, 4], t[2, 4], t[3, 4], t[1, 4]] - \\
& 6 KZI[b, b, a, a] P[t[1, 4], t[2, 4], t[3, 4], t[1, 4]] - \\
& 4 KZI[b, a, a, a] P[t[1, 4], t[2, 4], t[3, 4], t[2, 4]] - \\
& KZI[b, a, a, b] P[t[1, 4], t[2, 4], t[3, 4], t[2, 4]] - \\
& 2 KZI[b, a, b, a] P[t[1, 4], t[2, 4], t[3, 4], t[2, 4]] - \\
& 3 KZI[b, b, a, a] P[t[1, 4], t[2, 4], t[3, 4], t[2, 4]] - \\
& KZI[b, b, a, b] P[t[1, 4], t[2, 4], t[3, 4], t[2, 4]] - \\
& 3 KZI[b, a, a, a] P[t[1, 4], t[2, 4], t[3, 4], t[3, 4]] + \\
& 2 KZI[b, a, a, b] P[t[1, 4], t[2, 4], t[3, 4], t[3, 4]] - \\
& 2 KZI[b, b, a, a] P[t[1, 4], t[2, 4], t[3, 4], t[3, 4]] + \\
& 2 KZI[b, b, a, b] P[t[1, 4], t[2, 4], t[3, 4], t[3, 4]] + \\
& 6 KZI[b, b, b, a] P[t[1, 4], t[2, 4], t[3, 4], t[3, 4]] - \\
& KZI[b, a, a, a] P[t[1, 4], t[3, 4], t[1, 4], t[2, 4]] - \\
& 4 KZI[b, b, a, a] P[t[1, 4], t[3, 4], t[1, 4], t[2, 4]] + \\
& 3 KZI[b, a, a, a] P[t[1, 4], t[3, 4], t[2, 4], t[1, 4]] + \\
& 2 KZI[b, a, b, a] P[t[1, 4], t[3, 4], t[2, 4], t[1, 4]] + \\
& 6 KZI[b, b, a, a] P[t[1, 4], t[3, 4], t[2, 4], t[1, 4]] + \\
& 2 KZI[b, a, a, a] P[t[1, 4], t[3, 4], t[2, 4], t[2, 4]] + \\
& KZI[b, a, b, a] P[t[1, 4], t[3, 4], t[2, 4], t[2, 4]] + \\
& KZI[b, b, a, a] P[t[1, 4], t[3, 4], t[2, 4], t[2, 4]] - \\
& KZI[b, b, b, a] P[t[1, 4], t[3, 4], t[2, 4], t[2, 4]] + \\
& 3 KZI[b, a, a, a] P[t[1, 4], t[3, 4], t[2, 4], t[3, 4]] + \\
& KZI[b, a, a, b] P[t[1, 4], t[3, 4], t[2, 4], t[3, 4]] + \\
& KZI[b, b, a, a] P[t[1, 4], t[3, 4], t[2, 4], t[3, 4]] +
\end{aligned}$$

$$\begin{aligned}
& \text{KZI}[b, b, a, b] P[t[1, 4], t[3, 4], t[2, 4], t[3, 4]] - \\
& \text{KZI}[b, b, b, a] P[t[1, 4], t[3, 4], t[2, 4], t[3, 4]] - \\
& \text{KZI}[b, a, a, a] P[t[1, 4], t[3, 4], t[3, 4], t[2, 4]] - \\
& \text{KZI}[b, a, a, b] P[t[1, 4], t[3, 4], t[3, 4], t[2, 4]] - \\
& \text{KZI}[b, b, a, a] P[t[1, 4], t[3, 4], t[3, 4], t[2, 4]] - \\
& \text{KZI}[b, b, a, b] P[t[1, 4], t[3, 4], t[3, 4], t[2, 4]] - \\
& \text{KZI}[b, b, b, a] P[t[1, 4], t[3, 4], t[3, 4], t[2, 4]] + \\
& \text{KZI}[b, a, a, a] P[t[2, 4], t[1, 4], t[1, 4], t[3, 4]] + \\
& \text{KZI}[b, a, b, a] P[t[2, 4], t[1, 4], t[1, 4], t[3, 4]] + \\
& \text{KZI}[b, b, a, a] P[t[2, 4], t[1, 4], t[1, 4], t[3, 4]] + \\
& 4 \text{KZI}[b, a, a, a] P[t[2, 4], t[1, 4], t[2, 4], t[3, 4]] - \\
& 3 \text{KZI}[b, a, a, b] P[t[2, 4], t[1, 4], t[2, 4], t[3, 4]] + \\
& 2 \text{KZI}[b, a, b, a] P[t[2, 4], t[1, 4], t[2, 4], t[3, 4]] + \\
& 7 \text{KZI}[b, b, a, a] P[t[2, 4], t[1, 4], t[2, 4], t[3, 4]] - \\
& 3 \text{KZI}[b, b, a, b] P[t[2, 4], t[1, 4], t[2, 4], t[3, 4]] - \\
& 6 \text{KZI}[b, b, b, a] P[t[2, 4], t[1, 4], t[2, 4], t[3, 4]] + \\
& \text{KZI}[b, a, a, a] P[t[2, 4], t[1, 4], t[3, 4], t[1, 4]] + \\
& 4 \text{KZI}[b, b, a, a] P[t[2, 4], t[1, 4], t[3, 4], t[1, 4]] + \\
& \text{KZI}[b, a, a, b] P[t[2, 4], t[1, 4], t[3, 4], t[2, 4]] + \\
& \text{KZI}[b, b, a, a] P[t[2, 4], t[1, 4], t[3, 4], t[2, 4]] + \\
& \text{KZI}[b, b, a, b] P[t[2, 4], t[1, 4], t[3, 4], t[2, 4]] + \\
& 2 \text{KZI}[b, b, b, a] P[t[2, 4], t[1, 4], t[3, 4], t[2, 4]] + \\
& \text{KZI}[b, a, a, a] P[t[2, 4], t[1, 4], t[3, 4], t[3, 4]] - \\
& 2 \text{KZI}[b, a, a, b] P[t[2, 4], t[1, 4], t[3, 4], t[3, 4]] + \\
& 2 \text{KZI}[b, b, a, a] P[t[2, 4], t[1, 4], t[3, 4], t[3, 4]] - \\
& 2 \text{KZI}[b, b, a, b] P[t[2, 4], t[1, 4], t[3, 4], t[3, 4]] - \\
& 4 \text{KZI}[b, b, b, a] P[t[2, 4], t[1, 4], t[3, 4], t[3, 4]] - \\
& 2 \text{KZI}[b, a, a, a] P[t[2, 4], t[2, 4], t[1, 4], t[3, 4]] + \\
& \text{KZI}[b, a, a, b] P[t[2, 4], t[2, 4], t[1, 4], t[3, 4]] - \\
& \text{KZI}[b, a, b, a] P[t[2, 4], t[2, 4], t[1, 4], t[3, 4]] - \\
& 4 \text{KZI}[b, b, a, a] P[t[2, 4], t[2, 4], t[1, 4], t[3, 4]] + \\
& \text{KZI}[b, b, a, b] P[t[2, 4], t[2, 4], t[1, 4], t[3, 4]] + \\
& 2 \text{KZI}[b, b, b, a] P[t[2, 4], t[2, 4], t[1, 4], t[3, 4]] - \\
& 2 \text{KZI}[b, a, a, a] P[t[2, 4], t[2, 4], t[3, 4], t[1, 4]] - \\
& \text{KZI}[b, a, b, a] P[t[2, 4], t[2, 4], t[3, 4], t[1, 4]] - \\
& \text{KZI}[b, b, a, a] P[t[2, 4], t[2, 4], t[3, 4], t[1, 4]] + \\
& \text{KZI}[b, b, b, a] P[t[2, 4], t[2, 4], t[3, 4], t[1, 4]] + \\
& \text{KZI}[b, a, a, a] P[t[2, 4], t[3, 4], t[1, 4], t[1, 4]] + \\
& \text{KZI}[b, a, b, a] P[t[2, 4], t[3, 4], t[1, 4], t[1, 4]] + \\
& \text{KZI}[b, b, a, a] P[t[2, 4], t[3, 4], t[1, 4], t[1, 4]] - \\
& \text{KZI}[b, a, a, b] P[t[2, 4], t[3, 4], t[1, 4], t[2, 4]] - \\
& \text{KZI}[b, b, a, a] P[t[2, 4], t[3, 4], t[1, 4], t[2, 4]] - \\
& \text{KZI}[b, b, a, b] P[t[2, 4], t[3, 4], t[1, 4], t[2, 4]] -
\end{aligned}$$

$$\begin{aligned}
& 2 \text{KZI}[b, b, b, a] P[t[2, 4], t[3, 4], t[1, 4], t[2, 4]] + \\
& \text{KZI}[b, a, a, a] P[t[2, 4], t[3, 4], t[1, 4], t[3, 4]] - \\
& \text{KZI}[b, a, a, b] P[t[2, 4], t[3, 4], t[1, 4], t[3, 4]] - \\
& \text{KZI}[b, b, a, a] P[t[2, 4], t[3, 4], t[1, 4], t[3, 4]] - \\
& \text{KZI}[b, b, a, b] P[t[2, 4], t[3, 4], t[1, 4], t[3, 4]] - \\
& 3 \text{KZI}[b, b, b, a] P[t[2, 4], t[3, 4], t[1, 4], t[3, 4]] + \\
& 4 \text{KZI}[b, a, a, a] P[t[2, 4], t[3, 4], t[2, 4], t[1, 4]] + \\
& \text{KZI}[b, a, a, b] P[t[2, 4], t[3, 4], t[2, 4], t[1, 4]] + \\
& 2 \text{KZI}[b, a, b, a] P[t[2, 4], t[3, 4], t[2, 4], t[1, 4]] + \\
& 3 \text{KZI}[b, b, a, a] P[t[2, 4], t[3, 4], t[2, 4], t[1, 4]] + \\
& \text{KZI}[b, b, a, b] P[t[2, 4], t[3, 4], t[2, 4], t[1, 4]] + \\
& \text{KZI}[b, a, a, a] P[t[2, 4], t[3, 4], t[3, 4], t[1, 4]] + \\
& \text{KZI}[b, a, a, b] P[t[2, 4], t[3, 4], t[3, 4], t[1, 4]] + \\
& \text{KZI}[b, b, a, a] P[t[2, 4], t[3, 4], t[3, 4], t[1, 4]] + \\
& \text{KZI}[b, b, a, b] P[t[2, 4], t[3, 4], t[3, 4], t[1, 4]] + \\
& \text{KZI}[b, b, b, a] P[t[2, 4], t[3, 4], t[3, 4], t[1, 4]] - \\
& \text{KZI}[b, a, a, a] P[t[3, 4], t[1, 4], t[1, 4], t[2, 4]] - \\
& \text{KZI}[b, a, b, a] P[t[3, 4], t[1, 4], t[1, 4], t[2, 4]] - \\
& \text{KZI}[b, b, a, a] P[t[3, 4], t[1, 4], t[1, 4], t[2, 4]] + \\
& 3 \text{KZI}[b, a, a, a] P[t[3, 4], t[1, 4], t[2, 4], t[1, 4]] + \\
& 2 \text{KZI}[b, a, b, a] P[t[3, 4], t[1, 4], t[2, 4], t[1, 4]] + \\
& 6 \text{KZI}[b, b, a, a] P[t[3, 4], t[1, 4], t[2, 4], t[1, 4]] + \\
& 2 \text{KZI}[b, a, a, a] P[t[3, 4], t[1, 4], t[2, 4], t[2, 4]] - \\
& \text{KZI}[b, a, a, b] P[t[3, 4], t[1, 4], t[2, 4], t[2, 4]] + \\
& \text{KZI}[b, a, b, a] P[t[3, 4], t[1, 4], t[2, 4], t[2, 4]] + \\
& 4 \text{KZI}[b, b, a, a] P[t[3, 4], t[1, 4], t[2, 4], t[2, 4]] - \\
& \text{KZI}[b, b, a, b] P[t[3, 4], t[1, 4], t[2, 4], t[2, 4]] - \\
& 2 \text{KZI}[b, b, b, a] P[t[3, 4], t[1, 4], t[2, 4], t[2, 4]] + \\
& 3 \text{KZI}[b, a, a, a] P[t[3, 4], t[1, 4], t[2, 4], t[3, 4]] - \\
& 5 \text{KZI}[b, a, a, b] P[t[3, 4], t[1, 4], t[2, 4], t[3, 4]] + \\
& 3 \text{KZI}[b, b, a, a] P[t[3, 4], t[1, 4], t[2, 4], t[3, 4]] - \\
& 5 \text{KZI}[b, b, a, b] P[t[3, 4], t[1, 4], t[2, 4], t[3, 4]] - \\
& 11 \text{KZI}[b, b, b, a] P[t[3, 4], t[1, 4], t[2, 4], t[3, 4]] - \\
& \text{KZI}[b, a, a, a] P[t[3, 4], t[1, 4], t[3, 4], t[2, 4]] + \\
& \text{KZI}[b, a, a, b] P[t[3, 4], t[1, 4], t[3, 4], t[2, 4]] + \\
& \text{KZI}[b, b, a, a] P[t[3, 4], t[1, 4], t[3, 4], t[2, 4]] + \\
& \text{KZI}[b, b, a, b] P[t[3, 4], t[1, 4], t[3, 4], t[2, 4]] + \\
& 3 \text{KZI}[b, b, b, a] P[t[3, 4], t[1, 4], t[3, 4], t[2, 4]] - \\
& 3 \text{KZI}[b, a, a, a] P[t[3, 4], t[2, 4], t[1, 4], t[1, 4]] - \\
& 2 \text{KZI}[b, a, b, a] P[t[3, 4], t[2, 4], t[1, 4], t[1, 4]] - \\
& 6 \text{KZI}[b, b, a, a] P[t[3, 4], t[2, 4], t[1, 4], t[1, 4]] - \\
& 4 \text{KZI}[b, a, a, a] P[t[3, 4], t[2, 4], t[1, 4], t[2, 4]] + \\
& 3 \text{KZI}[b, a, a, b] P[t[3, 4], t[2, 4], t[1, 4], t[2, 4]] -
\end{aligned}$$

2 KZI[b, a, b, a] P[t[3, 4], t[2, 4], t[1, 4], t[2, 4]] -
 7 KZI[b, b, a, a] P[t[3, 4], t[2, 4], t[1, 4], t[2, 4]] +
 3 KZI[b, b, a, b] P[t[3, 4], t[2, 4], t[1, 4], t[2, 4]] +
 6 KZI[b, b, b, a] P[t[3, 4], t[2, 4], t[1, 4], t[2, 4]] -
 3 KZI[b, a, a, a] P[t[3, 4], t[2, 4], t[1, 4], t[3, 4]] +
 5 KZI[b, a, a, b] P[t[3, 4], t[2, 4], t[1, 4], t[3, 4]] -
 3 KZI[b, b, a, a] P[t[3, 4], t[2, 4], t[1, 4], t[3, 4]] +
 5 KZI[b, b, a, b] P[t[3, 4], t[2, 4], t[1, 4], t[3, 4]] +
 11 KZI[b, b, b, a] P[t[3, 4], t[2, 4], t[1, 4], t[3, 4]] -
 2 KZI[b, a, a, b] P[t[3, 4], t[2, 4], t[2, 4], t[1, 4]] +
 2 KZI[b, b, a, a] P[t[3, 4], t[2, 4], t[2, 4], t[1, 4]] -
 2 KZI[b, b, a, b] P[t[3, 4], t[2, 4], t[2, 4], t[1, 4]] -
 3 KZI[b, b, b, a] P[t[3, 4], t[2, 4], t[2, 4], t[1, 4]] -
 3 KZI[b, a, a, a] P[t[3, 4], t[2, 4], t[3, 4], t[1, 4]] -
 KZI[b, a, a, b] P[t[3, 4], t[2, 4], t[3, 4], t[1, 4]] -
 KZI[b, b, a, a] P[t[3, 4], t[2, 4], t[3, 4], t[1, 4]] -
 KZI[b, b, a, b] P[t[3, 4], t[2, 4], t[3, 4], t[1, 4]] +
 KZI[b, b, b, a] P[t[3, 4], t[2, 4], t[3, 4], t[1, 4]] -
 KZI[b, a, a, a] P[t[3, 4], t[3, 4], t[1, 4], t[2, 4]] +
 2 KZI[b, a, a, b] P[t[3, 4], t[3, 4], t[1, 4], t[2, 4]] -
 2 KZI[b, b, a, a] P[t[3, 4], t[3, 4], t[1, 4], t[2, 4]] +
 2 KZI[b, b, a, b] P[t[3, 4], t[3, 4], t[1, 4], t[2, 4]] +
 4 KZI[b, b, b, a] P[t[3, 4], t[3, 4], t[1, 4], t[2, 4]] +
 3 KZI[b, a, a, a] P[t[3, 4], t[3, 4], t[2, 4], t[1, 4]] -
 2 KZI[b, a, a, b] P[t[3, 4], t[3, 4], t[2, 4], t[1, 4]] +
 2 KZI[b, b, a, a] P[t[3, 4], t[3, 4], t[2, 4], t[1, 4]] -
 2 KZI[b, b, a, b] P[t[3, 4], t[3, 4], t[2, 4], t[1, 4]] -
 6 KZI[b, b, b, a] P[t[3, 4], t[3, 4], t[2, 4], t[1, 4]]

Cases[t2, KZI[___, b] | KZI[a, ___], Infinity]

{KZI[b, a, a, b], KZI[b, b, a, b], KZI[b, a, a, b], KZI[b, b, a, b],
 KZI[b, a, a, b], KZI[b, b, a, b], KZI[b, a, a, b], KZI[b, b, a, b], KZI[b, a, a, b],
 KZI[b, b, a, b], KZI[b, a, a, b], KZI[b, b, a, b], KZI[b, a, a, b], KZI[b, b, a, b],
 KZI[b, a, a, b], KZI[b, b, a, b], KZI[b, a, a, b], KZI[b, b, a, b], KZI[b, a, a, b],
 KZI[b, b, a, b], KZI[b, a, a, b], KZI[b, b, a, b], KZI[b, a, a, b], KZI[b, b, a, b],
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 KZI[b, b, a, b], KZI[b, a, a, b], KZI[b, b, a, b], KZI[b, a, a, b], KZI[b, b, a, b]}