

The HOMFLY Braidor Algebra - Further Computations

First Load "Programs"!

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fs = FormalExp[8, 2, P[{1, 2}, 2]] ** FormalExp[8, 2, P[{1, 2}, 1]]
ASeries[2, P[{1, 2}], P[{1, 2}, 1] + P[{1, 2}, 2],
  -x P[{2, 1}, 1] + x P[{2, 1}, 2] +  $\frac{1}{2}$  P[{1, 2}, 1, 1] + P[{1, 2}, 1, 2] +  $\frac{1}{2}$  P[{1, 2}, 2, 2],
  -x P[{2, 1}, 1, 1] + x P[{2, 1}, 2, 2] +  $\frac{1}{6}$  P[{1, 2}, 1, 1, 1] +
   $\frac{1}{2}$  P[{1, 2}, 1, 1, 2] +  $\frac{1}{2}$  P[{1, 2}, 1, 2, 2] +  $\frac{1}{6}$  P[{1, 2}, 2, 2, 2],
  - $\frac{7}{12}$  x P[{2, 1}, 1, 1, 1] -  $\frac{1}{4}$  x P[{2, 1}, 1, 1, 2] +  $\frac{1}{4}$  x P[{2, 1}, 1, 2, 2] +
   $\frac{7}{12}$  x P[{2, 1}, 2, 2, 2] +  $\frac{1}{24}$  P[{1, 2}, 1, 1, 1, 1] +  $\frac{1}{6}$  P[{1, 2}, 1, 1, 1, 2] +
   $\frac{1}{4}$  P[{1, 2}, 1, 1, 2, 2] +  $\frac{1}{6}$  P[{1, 2}, 1, 2, 2, 2] +  $\frac{1}{24}$  P[{1, 2}, 2, 2, 2, 2],
  - $\frac{1}{4}$  x P[{2, 1}, 1, 1, 1, 1] -  $\frac{1}{6}$  x P[{2, 1}, 1, 1, 1, 2] +  $\frac{1}{6}$  x P[{2, 1}, 1, 2, 2, 2] +
   $\frac{1}{4}$  x P[{2, 1}, 2, 2, 2, 2] +  $\frac{1}{120}$  P[{1, 2}, 1, 1, 1, 1, 1] +  $\frac{1}{24}$  P[{1, 2}, 1, 1, 1, 1, 2] +
   $\frac{1}{12}$  P[{1, 2}, 1, 1, 1, 2, 2] +  $\frac{1}{12}$  P[{1, 2}, 1, 1, 2, 2, 2] +
   $\frac{1}{24}$  P[{1, 2}, 1, 2, 2, 2, 2] +  $\frac{1}{120}$  P[{1, 2}, 2, 2, 2, 2, 2],
  - $\frac{31}{360}$  x P[{2, 1}, 1, 1, 1, 1, 1] -  $\frac{5}{72}$  x P[{2, 1}, 1, 1, 1, 1, 2] -  $\frac{1}{36}$  x P[{2, 1}, 1, 1, 1, 2, 2] +
   $\frac{1}{36}$  x P[{2, 1}, 1, 1, 2, 2, 2] +  $\frac{5}{72}$  x P[{2, 1}, 1, 2, 2, 2, 2] +  $\frac{31}{360}$  x P[{2, 1}, 2, 2, 2, 2, 2] +
   $\frac{1}{720}$  P[{1, 2}, 1, 1, 1, 1, 1, 1] +  $\frac{1}{120}$  P[{1, 2}, 1, 1, 1, 1, 1, 2] +
   $\frac{1}{48}$  P[{1, 2}, 1, 1, 1, 1, 2, 2] +  $\frac{1}{36}$  P[{1, 2}, 1, 1, 1, 2, 2, 2] +  $\frac{1}{48}$  P[{1, 2}, 1, 1, 2, 2, 2, 2] +
   $\frac{1}{120}$  P[{1, 2}, 1, 2, 2, 2, 2, 2] +  $\frac{1}{720}$  P[{1, 2}, 2, 2, 2, 2, 2, 2],
  - $\frac{1}{40}$  x P[{2, 1}, 1, 1, 1, 1, 1, 1] -  $\frac{1}{45}$  x P[{2, 1}, 1, 1, 1, 1, 1, 2] -
   $\frac{1}{72}$  x P[{2, 1}, 1, 1, 1, 1, 2, 2] +  $\frac{1}{72}$  x P[{2, 1}, 1, 1, 2, 2, 2, 2] +
   $\frac{1}{45}$  x P[{2, 1}, 1, 2, 2, 2, 2, 2] +  $\frac{1}{40}$  x P[{2, 1}, 2, 2, 2, 2, 2, 2] +
   $\frac{P[{1, 2}, 1, 1, 1, 1, 1, 1, 1]}{5040}$  +  $\frac{1}{720}$  P[{1, 2}, 1, 1, 1, 1, 1, 1, 2] +
   $\frac{1}{240}$  P[{1, 2}, 1, 1, 1, 1, 1, 2, 2] +  $\frac{1}{144}$  P[{1, 2}, 1, 1, 1, 1, 2, 2, 2] +
   $\frac{1}{144}$  P[{1, 2}, 1, 1, 1, 2, 2, 2, 2] +  $\frac{1}{240}$  P[{1, 2}, 1, 1, 2, 2, 2, 2, 2] +

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$$\begin{aligned} & \frac{1}{720} P[\{1, 2\}, 1, 2, 2, 2, 2, 2, 2] + \frac{P[\{1, 2\}, 2, 2, 2, 2, 2, 2, 2]}{5040}, \\ & - \frac{127 \times P[\{2, 1\}, 1, 1, 1, 1, 1, 1, 1]}{20160} - \frac{17 \times P[\{2, 1\}, 1, 1, 1, 1, 1, 1, 2]}{2880} - \\ & \frac{13 \times P[\{2, 1\}, 1, 1, 1, 1, 1, 2, 2]}{2880} - \frac{1}{576} \times P[\{2, 1\}, 1, 1, 1, 1, 2, 2, 2] + \\ & \frac{1}{576} \times P[\{2, 1\}, 1, 1, 1, 2, 2, 2, 2] + \frac{13 \times P[\{2, 1\}, 1, 1, 2, 2, 2, 2, 2]}{2880} + \\ & \frac{17 \times P[\{2, 1\}, 1, 2, 2, 2, 2, 2, 2]}{2880} + \frac{127 \times P[\{2, 1\}, 2, 2, 2, 2, 2, 2, 2]}{20160} + \\ & \frac{P[\{1, 2\}, 1, 1, 1, 1, 1, 1, 1]}{40320} + \frac{P[\{1, 2\}, 1, 1, 1, 1, 1, 1, 2]}{5040} + \\ & \frac{P[\{1, 2\}, 1, 1, 1, 1, 1, 2, 2]}{1440} + \frac{1}{720} P[\{1, 2\}, 1, 1, 1, 1, 1, 2, 2, 2] + \\ & \frac{1}{576} P[\{1, 2\}, 1, 1, 1, 1, 2, 2, 2, 2] + \\ & \frac{1}{720} P[\{1, 2\}, 1, 1, 1, 2, 2, 2, 2, 2] + \frac{P[\{1, 2\}, 1, 1, 2, 2, 2, 2, 2, 2]}{1440} + \\ & \frac{P[\{1, 2\}, 1, 2, 2, 2, 2, 2, 2, 2]}{5040} + \frac{P[\{1, 2\}, 2, 2, 2, 2, 2, 2, 2]}{40320} \end{aligned}$$

GeneratingSeries[\{1, 2\}, fs]

$$\begin{aligned} & \text{ASeries}[2, 1, t[1] + t[2], \frac{t[1]^2}{2} + t[1] t[2] + \frac{t[2]^2}{2}, \\ & \frac{t[1]^3}{6} + \frac{1}{2} t[1]^2 t[2] + \frac{1}{2} t[1] t[2]^2 + \frac{t[2]^3}{6}, \\ & \frac{t[1]^4}{24} + \frac{1}{6} t[1]^3 t[2] + \frac{1}{4} t[1]^2 t[2]^2 + \frac{1}{6} t[1] t[2]^3 + \frac{t[2]^4}{24}, \\ & \frac{t[1]^5}{120} + \frac{1}{24} t[1]^4 t[2] + \frac{1}{12} t[1]^3 t[2]^2 + \frac{1}{12} t[1]^2 t[2]^3 + \frac{1}{24} t[1] t[2]^4 + \frac{t[2]^5}{120}, \\ & \frac{t[1]^6}{720} + \frac{1}{120} t[1]^5 t[2] + \frac{1}{48} t[1]^4 t[2]^2 + \frac{1}{36} t[1]^3 t[2]^3 + \frac{1}{48} t[1]^2 t[2]^4 + \frac{1}{120} t[1] t[2]^5 + \\ & \frac{t[2]^6}{720}, \frac{t[1]^7}{5040} + \frac{1}{720} t[1]^6 t[2] + \frac{1}{240} t[1]^5 t[2]^2 + \frac{1}{144} t[1]^4 t[2]^3 + \frac{1}{144} t[1]^3 t[2]^4 + \\ & \frac{1}{240} t[1]^2 t[2]^5 + \frac{1}{720} t[1] t[2]^6 + \frac{t[2]^7}{5040}, \frac{t[1]^8}{40320} + \frac{t[1]^7 t[2]}{5040} + \frac{t[1]^6 t[2]^2}{1440} + \\ & \frac{1}{720} t[1]^5 t[2]^3 + \frac{1}{576} t[1]^4 t[2]^4 + \frac{1}{720} t[1]^3 t[2]^5 + \frac{t[1]^2 t[2]^6}{1440} + \frac{t[1] t[2]^7}{5040} + \frac{t[2]^8}{40320}] \end{aligned}$$

gs2 = GeneratingSeries[\{2, 1\}, fs]

$$\begin{aligned} & \text{ASeries}[2, 0, 0, -x t[1] + x t[2], -x t[1]^2 + x t[2]^2, \\ & - \frac{7}{12} x t[1]^3 - \frac{1}{4} x t[1]^2 t[2] + \frac{1}{4} x t[1] t[2]^2 + \frac{7}{12} x t[2]^3, \\ & - \frac{1}{4} x t[1]^4 - \frac{1}{6} x t[1]^3 t[2] + \frac{1}{6} x t[1] t[2]^3 + \frac{1}{4} x t[2]^4, - \frac{31}{360} x t[1]^5 - \\ & \frac{5}{72} x t[1]^4 t[2] - \frac{1}{36} x t[1]^3 t[2]^2 + \frac{1}{36} x t[1]^2 t[2]^3 + \frac{5}{72} x t[1] t[2]^4 + \frac{31}{360} x t[2]^5, \\ & - \frac{1}{40} x t[1]^6 - \frac{1}{45} x t[1]^5 t[2] - \frac{1}{72} x t[1]^4 t[2]^2 + \frac{1}{72} x t[1]^2 t[2]^4 + \frac{1}{45} x t[1] t[2]^5 + \frac{1}{40} x t[2]^6, \\ & - \frac{127 x t[1]^7}{20160} - \frac{17 x t[1]^6 t[2]}{2880} - \frac{13 x t[1]^5 t[2]^2}{2880} - \frac{1}{576} x t[1]^4 t[2]^3 + \\ & \frac{1}{576} x t[1]^3 t[2]^4 + \frac{13 x t[1]^2 t[2]^5}{2880} + \frac{17 x t[1] t[2]^6}{2880} + \frac{127 x t[2]^7}{20160}] \end{aligned}$$

GeneratingSeries [{1, 2}, **FormalExp**[8, 2, **P**[{1, 2}, 1]] ** **FormalExp**[8, 2, **P**[{1, 2}, 2]]]

$$\begin{aligned} & \text{ASeries}[2, 1, t[1] + t[2], \frac{t[1]^2}{2} + t[1] t[2] + \frac{t[2]^2}{2}, \\ & \frac{t[1]^3}{6} + \frac{1}{2} t[1]^2 t[2] + \frac{1}{2} t[1] t[2]^2 + \frac{t[2]^3}{6}, \\ & \frac{t[1]^4}{24} + \frac{1}{6} t[1]^3 t[2] + \frac{1}{4} t[1]^2 t[2]^2 + \frac{1}{6} t[1] t[2]^3 + \frac{t[2]^4}{24}, \\ & \frac{t[1]^5}{120} + \frac{1}{24} t[1]^4 t[2] + \frac{1}{12} t[1]^3 t[2]^2 + \frac{1}{12} t[1]^2 t[2]^3 + \frac{1}{24} t[1] t[2]^4 + \frac{t[2]^5}{120}, \\ & \frac{t[1]^6}{720} + \frac{1}{120} t[1]^5 t[2] + \frac{1}{48} t[1]^4 t[2]^2 + \frac{1}{36} t[1]^3 t[2]^3 + \frac{1}{48} t[1]^2 t[2]^4 + \frac{1}{120} t[1] t[2]^5 + \\ & \frac{t[2]^6}{720}, \frac{t[1]^7}{5040} + \frac{1}{720} t[1]^6 t[2] + \frac{1}{240} t[1]^5 t[2]^2 + \frac{1}{144} t[1]^4 t[2]^3 + \frac{1}{144} t[1]^3 t[2]^4 + \\ & \frac{1}{240} t[1]^2 t[2]^5 + \frac{1}{720} t[1] t[2]^6 + \frac{t[2]^7}{5040}, \frac{t[1]^8}{40320} + \frac{t[1]^7 t[2]}{5040} + \frac{t[1]^6 t[2]^2}{1440} + \\ & \frac{1}{720} t[1]^5 t[2]^3 + \frac{1}{576} t[1]^4 t[2]^4 + \frac{1}{720} t[1]^3 t[2]^5 + \frac{t[1]^2 t[2]^6}{1440} + \frac{t[1] t[2]^7}{5040} + \frac{t[2]^8}{40320}] \end{aligned}$$

CoefficientList[**gs2** /. {**x** → -1/(t[1] - t[2]), t[i_] ⇒ ht[i]}, **h**] *
 {1/2, 1, 3, 12, 60, 360, 2520, 20160} // **Factor** // **Expand**

Thread::tdlen: Objects of unequal length in

$$\begin{aligned} & \{ \text{ASeries}[2, 0, \langle\langle 6 \rangle\rangle, \frac{h^6 t[1]^6}{40 (t[\langle\langle 1 \rangle\rangle] + \text{Times}[\langle\langle 2 \rangle\rangle])} + \frac{h^6 \langle\langle 1 \rangle\rangle^5 t[2]}{45 \langle\langle 1 \rangle\rangle} + \langle\langle 1 \rangle\rangle - \langle\langle 1 \rangle\rangle - \frac{\langle\langle 1 \rangle\rangle}{45 \langle\langle 1 \rangle\rangle} - \\ & \frac{h^6 t[2]^6}{40 \langle\langle 1 \rangle\rangle}, \frac{127 h^7 t[1]^7}{20160 (t[\langle\langle 1 \rangle\rangle] + \text{Times}[\langle\langle 2 \rangle\rangle])} + \frac{17 h^7 t[1]^6 t[2]}{2880 (t[\langle\langle 1 \rangle\rangle] + \text{Times}[\langle\langle 2 \rangle\rangle])} + \langle\langle 10 \rangle\rangle \} \langle\langle 1 \rangle\rangle \end{aligned}$$

cannot be combined. More...

Thread::tdlen: Objects of unequal length in {Integrate`V[1][6]} {1/2, 1, 3, 12, 60, 360, 2520, 20160} cannot be combined. More...

$$\begin{aligned} & \text{Factor} \left[\left\{ \text{ASeries}[2, 0, 0, \frac{h t[1]}{t[1] - t[2]} - \frac{h t[2]}{t[1] - t[2]}, \frac{h^2 t[1]^2}{t[1] - t[2]} - \frac{h^2 t[2]^2}{t[1] - t[2]}, \right. \right. \\ & \frac{7 h^3 t[1]^3}{12 (t[1] - t[2])} + \frac{h^3 t[1]^2 t[2]}{4 (t[1] - t[2])} - \frac{h^3 t[1] t[2]^2}{4 (t[1] - t[2])} - \frac{7 h^3 t[2]^3}{12 (t[1] - t[2])}, \\ & \frac{h^4 t[1]^4}{4 (t[1] - t[2])} + \frac{h^4 t[1]^3 t[2]}{6 (t[1] - t[2])} - \frac{h^4 t[1] t[2]^3}{6 (t[1] - t[2])} - \frac{h^4 t[2]^4}{4 (t[1] - t[2])}, \\ & \frac{31 h^5 t[1]^5}{360 (t[1] - t[2])} + \frac{5 h^5 t[1]^4 t[2]}{72 (t[1] - t[2])} + \frac{h^5 t[1]^3 t[2]^2}{36 (t[1] - t[2])} - \frac{h^5 t[1]^2 t[2]^3}{36 (t[1] - t[2])} - \\ & \frac{5 h^5 t[1] t[2]^4}{72 (t[1] - t[2])} - \frac{31 h^5 t[2]^5}{360 (t[1] - t[2])}, \frac{h^6 t[1]^6}{40 (t[1] - t[2])} + \frac{h^6 t[1]^5 t[2]}{45 (t[1] - t[2])} + \\ & \frac{h^6 t[1]^4 t[2]^2}{72 (t[1] - t[2])} - \frac{h^6 t[1]^2 t[2]^4}{72 (t[1] - t[2])} - \frac{h^6 t[1] t[2]^5}{45 (t[1] - t[2])} - \frac{h^6 t[2]^6}{40 (t[1] - t[2])}, \\ & \frac{127 h^7 t[1]^7}{20160 (t[1] - t[2])} + \frac{17 h^7 t[1]^6 t[2]}{2880 (t[1] - t[2])} + \frac{13 h^7 t[1]^5 t[2]^2}{2880 (t[1] - t[2])} + \frac{h^7 t[1]^4 t[2]^3}{576 (t[1] - t[2])} - \\ & \left. \left. \frac{h^7 t[1]^3 t[2]^4}{576 (t[1] - t[2])} - \frac{13 h^7 t[1]^2 t[2]^5}{2880 (t[1] - t[2])} - \frac{17 h^7 t[1] t[2]^6}{2880 (t[1] - t[2])} - \frac{127 h^7 t[2]^7}{20160 (t[1] - t[2])} \right\} \right] \\ & \left\{ \frac{1}{2}, 1, 3, 12, 60, 360, 2520, 20160 \right\} \end{aligned}$$

$$\begin{aligned} & (127 t[1]^6 + 246 t[1]^5 t[2] + 337 t[1]^4 t[2]^2 + 372 t[1]^3 t[2]^3 + \\ & 337 t[1]^2 t[2]^4 + 246 t[1] t[2]^5 + 127 t[2]^6) - \text{Expand}[127 (t[1] + t[2])^6] + \\ & \text{Expand}[516 t[1] t[2] (t[1] + t[2])^4] - \text{Expand}[496 t[1]^2 t[2]^2 (t[1] + t[2])^2] \\ & - 64 t[1]^3 t[2]^3 \end{aligned}$$

GeneratingSeries [{1, 2}, **FormalExp**[8, 2, **P** [{1, 2}, 1] + **P** [{1, 2}, 2]]]

$$\begin{aligned}
 & \text{ASeries}[2, 1, t[1] + t[2], \frac{t[1]^2}{2} + t[1] t[2] + \frac{t[2]^2}{2}, \\
 & \frac{1}{6} x^2 t[1] + \frac{t[1]^3}{6} - \frac{1}{6} x^2 t[2] + \frac{1}{2} t[1]^2 t[2] + \frac{1}{2} t[1] t[2]^2 + \frac{t[2]^3}{6}, \frac{1}{8} x^2 t[1]^2 + \frac{t[1]^4}{24} + \\
 & \frac{1}{12} x^2 t[1] t[2] + \frac{1}{6} t[1]^3 t[2] - \frac{5}{24} x^2 t[2]^2 + \frac{1}{4} t[1]^2 t[2]^2 + \frac{1}{6} t[1] t[2]^3 + \frac{t[2]^4}{24}, \\
 & \frac{1}{120} x^4 t[1] + \frac{1}{20} x^2 t[1]^3 + \frac{t[1]^5}{120} - \frac{1}{120} x^4 t[2] + \frac{1}{10} x^2 t[1]^2 t[2] + \frac{1}{24} t[1]^4 t[2] - \\
 & \frac{1}{60} x^2 t[1] t[2]^2 + \frac{1}{12} t[1]^3 t[2]^2 - \frac{2}{15} x^2 t[2]^3 + \frac{1}{12} t[1]^2 t[2]^3 + \frac{1}{24} t[1] t[2]^4 + \frac{t[2]^5}{120}, \\
 & \frac{1}{144} x^4 t[1]^2 + \frac{1}{72} x^2 t[1]^4 + \frac{t[1]^6}{720} + \frac{1}{360} x^4 t[1] t[2] + \frac{2}{45} x^2 t[1]^3 t[2] + \\
 & \frac{1}{120} t[1]^5 t[2] - \frac{7}{720} x^4 t[2]^2 + \frac{1}{30} x^2 t[1]^2 t[2]^2 + \frac{1}{48} t[1]^4 t[2]^2 - \frac{1}{30} x^2 t[1] t[2]^3 + \\
 & \frac{1}{36} t[1]^3 t[2]^3 - \frac{7}{120} x^2 t[2]^4 + \frac{1}{48} t[1]^2 t[2]^4 + \frac{1}{120} t[1] t[2]^5 + \frac{t[2]^6}{720}, \\
 & \frac{x^6 t[1]}{5040} + \frac{1}{336} x^4 t[1]^3 + \frac{1}{336} x^2 t[1]^5 + \frac{t[1]^7}{5040} - \frac{x^6 t[2]}{5040} + \frac{5 x^4 t[1]^2 t[2]}{1008} + \\
 & \frac{13 x^2 t[1]^4 t[2]}{1008} + \frac{1}{720} t[1]^6 t[2] - \frac{11 x^4 t[1] t[2]^2}{5040} + \frac{47 x^2 t[1]^3 t[2]^2}{2520} + \\
 & \frac{1}{240} t[1]^5 t[2]^2 - \frac{29 x^4 t[2]^3}{5040} + \frac{1}{280} x^2 t[1]^2 t[2]^3 + \frac{1}{144} t[1]^4 t[2]^3 - \frac{31 x^2 t[1] t[2]^4}{1680} + \\
 & \frac{1}{144} t[1]^3 t[2]^4 - \frac{11}{560} x^2 t[2]^5 + \frac{1}{240} t[1]^2 t[2]^5 + \frac{1}{720} t[1] t[2]^6 + \frac{t[2]^7}{5040}, \\
 & \frac{x^6 t[1]^2}{5760} + \frac{x^4 t[1]^4}{1152} + \frac{x^2 t[1]^6}{1920} + \frac{t[1]^8}{40320} + \frac{x^6 t[1] t[2]}{20160} + \frac{5 x^4 t[1]^3 t[2]}{2016} + \frac{19 x^2 t[1]^5 t[2]}{6720} + \\
 & \frac{t[1]^7 t[2]}{5040} - \frac{x^6 t[2]^2}{4480} + \frac{5 x^4 t[1]^2 t[2]^2}{4032} + \frac{47 x^2 t[1]^4 t[2]^2}{8064} + \frac{t[1]^6 t[2]^2}{1440} - \frac{23 x^4 t[1] t[2]^3}{10080} + \\
 & \frac{47 x^2 t[1]^3 t[2]^3}{10080} + \frac{1}{720} t[1]^5 t[2]^3 - \frac{31 x^4 t[2]^4}{13440} - \frac{23 x^2 t[1]^2 t[2]^4}{13440} + \frac{1}{576} t[1]^4 t[2]^4 - \\
 & \frac{3}{448} x^2 t[1] t[2]^5 + \frac{1}{720} t[1]^3 t[2]^5 - \frac{73 x^2 t[2]^6}{13440} + \frac{t[1]^2 t[2]^6}{1440} + \frac{t[1] t[2]^7}{5040} + \frac{t[2]^8}{40320}]
 \end{aligned}$$

GeneratingSeries [{2, 1}, **FormalExp**[8, 2, **P** [{1, 2}, 1] + **P** [{1, 2}, 2]]]

$$\begin{aligned}
 & \text{ASeries}\left[2, 0, 0, -\frac{1}{2} x t[1] + \frac{1}{2} x t[2], -\frac{1}{3} x t[1]^2 - \frac{1}{3} x t[1] t[2] + \frac{2}{3} x t[2]^2, \right. \\
 & -\frac{1}{24} x^3 t[1] - \frac{1}{8} x t[1]^3 + \frac{1}{24} x^3 t[2] - \frac{7}{24} x t[1]^2 t[2] - \frac{1}{24} x t[1] t[2]^2 + \frac{11}{24} x t[2]^3, \\
 & -\frac{1}{30} x^3 t[1]^2 - \frac{1}{30} x t[1]^4 - \frac{1}{60} x^3 t[1] t[2] - \frac{7}{60} x t[1]^3 t[2] + \frac{1}{20} x^3 t[2]^2 - \\
 & \frac{7}{60} x t[1]^2 t[2]^2 + \frac{1}{20} x t[1] t[2]^3 + \frac{13}{60} x t[2]^4, -\frac{1}{720} x^5 t[1] - \frac{1}{72} x^3 t[1]^3 - \\
 & \frac{1}{144} x t[1]^5 + \frac{1}{720} x^5 t[2] - \frac{1}{40} x^3 t[1]^2 t[2] - \frac{23}{720} x t[1]^4 t[2] + \frac{1}{120} x^3 t[1] t[2]^2 - \\
 & \frac{19}{360} x t[1]^3 t[2]^2 + \frac{11}{360} x^3 t[2]^3 - \frac{1}{40} x t[1]^2 t[2]^3 + \frac{3}{80} x t[1] t[2]^4 + \frac{19}{240} x t[2]^5, \\
 & -\frac{1}{840} x^5 t[1]^2 - \frac{1}{252} x^3 t[1]^4 - \frac{1}{840} x t[1]^6 - \frac{x^5 t[1] t[2]}{2520} - \frac{1}{84} x^3 t[1]^3 t[2] - \\
 & \frac{17 x t[1]^5 t[2]}{2520} + \frac{1}{630} x^5 t[2]^2 - \frac{1}{140} x^3 t[1]^2 t[2]^2 - \frac{19 x t[1]^4 t[2]^2}{1260} + \frac{13 x^3 t[1] t[2]^3}{1260} - \\
 & \frac{19 x t[1]^3 t[2]^3}{1260} + \frac{4}{315} x^3 t[2]^4 - \frac{1}{840} x t[1]^2 t[2]^4 + \frac{13}{840} x t[1] t[2]^5 + \frac{1}{42} x t[2]^6, \\
 & -\frac{x^7 t[1]}{40320} - \frac{x^5 t[1]^3}{1920} - \frac{x^3 t[1]^5}{1152} - \frac{x t[1]^7}{5760} + \frac{x^7 t[2]}{40320} - \frac{11 x^5 t[1]^2 t[2]}{13440} - \frac{29 x^3 t[1]^4 t[2]}{8064} - \\
 & \frac{47 x t[1]^6 t[2]}{40320} + \frac{17 x^5 t[1] t[2]^2}{40320} - \frac{19 x^3 t[1]^3 t[2]^2}{4032} - \frac{131 x t[1]^5 t[2]^2}{40320} + \\
 & \frac{37 x^5 t[2]^3}{40320} - \frac{x^3 t[1]^2 t[2]^3}{20160} - \frac{187 x t[1]^4 t[2]^3}{40320} + \frac{209 x^3 t[1] t[2]^4}{40320} - \\
 & \left. \frac{13 x t[1]^3 t[2]^4}{4480} + \frac{163 x^3 t[2]^5}{40320} + \frac{17 x t[1]^2 t[2]^5}{13440} + \frac{191 x t[1] t[2]^6}{40320} + \frac{247 x t[2]^7}{40320} \right]
 \end{aligned}$$

{k, 1} = {7, 4}; **ps** = **FormalPower**[**P** [{1, 2}, 2], k] ** **FormalPower**[**P** [{1, 2}, 1], 1]

$$\begin{aligned}
 & -x P\{2, 1\}, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\} - \\
 & x P\{2, 1\}, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2\} - x P\{2, 1\}, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2\} - \\
 & x P\{2, 1\}, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2\} + x P\{2, 1\}, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2\} + \\
 & x P\{2, 1\}, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2\} + x P\{2, 1\}, 1, 2, 2, 2, 2, 2, 2, 2, 2, 2\} + \\
 & x P\{2, 1\}, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2\} + P\{1, 2\}, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2\}
 \end{aligned}$$

ps12 = **GeneratingSeries** [{1, 2}, **ps**] /. {t[1] → t1, t[2] → t2}

$$t1^4 t2^7$$

ps12 == t1^1 t2^k

True

Expand[**ps21** = **GeneratingSeries** [{2, 1}, **ps**] /. {t[1] → t1, t[2] → t2}]

$$-t1^{10} x - t1^9 t2 x - t1^8 t2^2 x - t1^7 t2^3 x + t1^3 t2^7 x + t1^2 t2^8 x + t1 t2^9 x + t2^{10} x$$

$$\text{ps21} == \text{Expand}\left[\text{Factor}\left[x \left(\frac{t2^{k+1} - t1^1 t2^k + t1^{k+1} - t1^k t2^1}{t2 - t1}\right)\right]\right]$$

True

FormalPower[**P** [{1, 2}, 2], 7] ** **FormalPower**[**P** [{1, 2}, 1], 1]

$$\begin{aligned}
 & -x P\{2, 1\}, 1, 1, 1, 1, 1, 1, 1\} + \\
 & x P\{2, 1\}, 2, 2, 2, 2, 2, 2, 2\} + P\{1, 2\}, 1, 2, 2, 2, 2, 2, 2\}
 \end{aligned}$$

FormalPower[P[{1, 2, 3}, 2], 3] **

FormalPower[P[{1, 2, 3}, 3], 4] ** **FormalPower**[P[{1, 2, 3}, 2], 1]

$$x^2 P[\{1, 2, 3\}, 2, 2, 2, 2, 2, 2] + x^2 P[\{1, 2, 3\}, 2, 2, 2, 2, 2, 3] + x^2 P[\{1, 2, 3\}, 2, 2, 2, 2, 3, 3] - x^2 P[\{1, 2, 3\}, 2, 2, 3, 3, 3, 3] - x^2 P[\{1, 2, 3\}, 2, 3, 3, 3, 3, 3] - x^2 P[\{1, 2, 3\}, 3, 3, 3, 3, 3, 3] - x P[\{1, 3, 2\}, 2, 2, 2, 2, 3, 3, 3] + x P[\{1, 3, 2\}, 3, 3, 3, 3, 3, 3, 3] + P[\{1, 2, 3\}, 2, 2, 2, 2, 3, 3, 3, 3]$$

d = 10; ps12 =

GeneratingSeries [{1, 2}, **Delta**[**FormalPower**[P[{1}, 1], **d**]]] /. {t[1] → t1, t[2] → t2}

$$t^{2^{10}} + t^{1^8} x^2 + 2 t^{1^7} t^2 x^2 + 3 t^{1^6} t^2 x^2 + 4 t^{1^5} t^2 x^2 + 5 t^{1^4} t^2 x^2 + 6 t^{1^3} t^2 x^2 + 7 t^{1^2} t^2 x^2 + 8 t^1 t^2 x^2 + 9 t^2 x^2 + t^{1^6} x^4 + 4 t^{1^5} t^2 x^4 + 10 t^{1^4} t^2 x^4 + 20 t^{1^3} t^2 x^4 + 35 t^{1^2} t^2 x^4 + 56 t^1 t^2 x^4 + 84 t^2 x^4 + t^{1^4} x^6 + 6 t^{1^3} t^2 x^6 + 21 t^{1^2} t^2 x^6 + 56 t^1 t^2 x^6 + 126 t^2 x^6 + t^{1^2} x^8 + 8 t^1 t^2 x^8 + 36 t^2 x^8 + x^{10}$$

CoefficientList[ps12, t1]

$$\{t^{2^{10}} + 9 t^2 x^2 + 84 t^2 x^4 + 126 t^2 x^6 + 36 t^2 x^8 + x^{10}, 8 t^2 x^2 + 56 t^2 x^4 + 56 t^2 x^6 + 8 t^2 x^8, 7 t^2 x^2 + 35 t^2 x^4 + 21 t^2 x^6 + x^8, 6 t^2 x^2 + 20 t^2 x^4 + 6 t^2 x^6, 5 t^2 x^2 + 10 t^2 x^4 + x^6, 4 t^2 x^2 + 4 t^2 x^4, 3 t^2 x^2 + x^4, 2 t^2 x^2, x^2\}$$

$$aps12 = \text{Expand}[t^d + x \sum_{k=0}^{d-1} \frac{1}{2} t^{d-1-k} ((t_2 + x)^k - (t_2 - x)^k)]$$

$$t^{2^{10}} + t^{1^8} x^2 + 2 t^{1^7} t^2 x^2 + 3 t^{1^6} t^2 x^2 + 4 t^{1^5} t^2 x^2 + 5 t^{1^4} t^2 x^2 + 6 t^{1^3} t^2 x^2 + 7 t^{1^2} t^2 x^2 + 8 t^1 t^2 x^2 + 9 t^2 x^2 + t^{1^6} x^4 + 4 t^{1^5} t^2 x^4 + 10 t^{1^4} t^2 x^4 + 20 t^{1^3} t^2 x^4 + 35 t^{1^2} t^2 x^4 + 56 t^1 t^2 x^4 + 84 t^2 x^4 + t^{1^4} x^6 + 6 t^{1^3} t^2 x^6 + 21 t^{1^2} t^2 x^6 + 56 t^1 t^2 x^6 + 126 t^2 x^6 + t^{1^2} x^8 + 8 t^1 t^2 x^8 + 36 t^2 x^8 + x^{10}$$

$$bps12 = \text{Expand}[\text{Factor}[t^d + \frac{x}{2} \left(\frac{(t_2 + x)^d - t_1^d}{t_2 + x - t_1} - \frac{(t_2 - x)^d - t_1^d}{t_2 - x - t_1} \right)]]$$

$$t^{2^{10}} + t^{1^8} x^2 + 2 t^{1^7} t^2 x^2 + 3 t^{1^6} t^2 x^2 + 4 t^{1^5} t^2 x^2 + 5 t^{1^4} t^2 x^2 + 6 t^{1^3} t^2 x^2 + 7 t^{1^2} t^2 x^2 + 8 t^1 t^2 x^2 + 9 t^2 x^2 + t^{1^6} x^4 + 4 t^{1^5} t^2 x^4 + 10 t^{1^4} t^2 x^4 + 20 t^{1^3} t^2 x^4 + 35 t^{1^2} t^2 x^4 + 56 t^1 t^2 x^4 + 84 t^2 x^4 + t^{1^4} x^6 + 6 t^{1^3} t^2 x^6 + 21 t^{1^2} t^2 x^6 + 56 t^1 t^2 x^6 + 126 t^2 x^6 + t^{1^2} x^8 + 8 t^1 t^2 x^8 + 36 t^2 x^8 + x^{10}$$

ps12 - bps12

0

CoefficientList[aps12, t1]

$$\{t^{2^9} + 8 t^2 x^2 + 56 t^2 x^4 + 56 t^2 x^6 + 8 t^2 x^8, 7 t^2 x^2 + 35 t^2 x^4 + 21 t^2 x^6 + x^8, 6 t^2 x^2 + 20 t^2 x^4 + 6 t^2 x^6, 5 t^2 x^2 + 10 t^2 x^4 + x^6, 4 t^2 x^2 + 4 t^2 x^4, 3 t^2 x^2 + x^4, 2 t^2 x^2, x^2\}$$

ps21 =

GeneratingSeries [{2, 1}, **Delta**[**FormalPower**[P[{1}, 1], **d**]]] /. {t[1] → t1, t[2] → t2}

$$t^{1^9} x + t^{1^8} t^2 x + t^{1^7} t^2 x + t^{1^6} t^2 x + t^{1^5} t^2 x + t^{1^4} t^2 x + t^{1^3} t^2 x + t^{1^2} t^2 x + t^1 t^2 x + t^2 x + t^{1^7} x^3 + 3 t^{1^6} t^2 x^3 + 6 t^{1^5} t^2 x^3 + 10 t^{1^4} t^2 x^3 + 15 t^{1^3} t^2 x^3 + 21 t^{1^2} t^2 x^3 + 28 t^1 t^2 x^3 + 36 t^2 x^3 + t^{1^5} x^5 + 5 t^{1^4} t^2 x^5 + 15 t^{1^3} t^2 x^5 + 35 t^{1^2} t^2 x^5 + 70 t^1 t^2 x^5 + 126 t^2 x^5 + t^{1^3} x^7 + 7 t^{1^2} t^2 x^7 + 28 t^1 t^2 x^7 + 84 t^2 x^7 + t^1 x^9 + 9 t^2 x^9$$

CoefficientList[ps21, t1]

$$\{t^9 x + 36 t^7 x^3 + 126 t^5 x^5 + 84 t^3 x^7 + 9 t^2 x^9, t^8 x + 28 t^6 x^3 + 70 t^4 x^5 + 28 t^2 x^7 + x^9, t^7 x + 21 t^5 x^3 + 35 t^3 x^5 + 7 t^2 x^7, t^6 x + 15 t^4 x^3 + 15 t^2 x^5 + x^7, t^5 x + 10 t^3 x^3 + 5 t^2 x^5, t^4 x + 6 t^2 x^3 + x^5, t^3 x + 3 t^2 x^3, t^2 x + x^3, t^2 x, x\}$$

$$\text{aps21} = \text{Expand}\left[x \sum_{k=0}^{d-1} \frac{t^{d-1-k}}{2} \left((t^2 + x)^k + (t^2 - x)^k \right)\right]$$

$$t^9 x + t^8 t^2 x + t^7 t^2 x + t^6 t^2 x + t^5 t^2 x + t^4 t^2 x + t^3 t^2 x + t^2 t^2 x + t^1 t^2 x + t^2 x + t^9 x + t^7 x^3 + 3 t^6 t^2 x^3 + 6 t^5 t^2 x^3 + 10 t^4 t^2 x^3 + 15 t^3 t^2 x^3 + 21 t^2 t^2 x^3 + 28 t^1 t^2 x^3 + 36 t^2 x^3 + t^5 x^5 + 5 t^4 t^2 x^5 + 15 t^3 t^2 x^5 + 35 t^1 t^2 x^5 + 70 t^1 t^2 x^5 + 126 t^5 x^5 + t^3 x^7 + 7 t^1 t^2 x^7 + 28 t^1 t^2 x^7 + 84 t^2 x^7 + t^1 x^9 + 9 t^2 x^9$$

$$\text{bps21} = \text{Expand}\left[\text{Factor}\left[\frac{x}{2} \left(\frac{(t^2 + x)^d - t^{1d}}{t^2 + x - t^1} + \frac{(t^2 - x)^d - t^{1d}}{t^2 - x - t^1} \right)\right]\right]$$

General::spell: Possible spelling error: new symbol name "bps21" is similar to existing symbols {aps21, bps12, ps21}. More...

$$t^9 x + t^8 t^2 x + t^7 t^2 x + t^6 t^2 x + t^5 t^2 x + t^4 t^2 x + t^3 t^2 x + t^2 t^2 x + t^1 t^2 x + t^2 x + t^9 x + t^7 x^3 + 3 t^6 t^2 x^3 + 6 t^5 t^2 x^3 + 10 t^4 t^2 x^3 + 15 t^3 t^2 x^3 + 21 t^2 t^2 x^3 + 28 t^1 t^2 x^3 + 36 t^2 x^3 + t^5 x^5 + 5 t^4 t^2 x^5 + 15 t^3 t^2 x^5 + 35 t^1 t^2 x^5 + 70 t^1 t^2 x^5 + 126 t^5 x^5 + t^3 x^7 + 7 t^1 t^2 x^7 + 28 t^1 t^2 x^7 + 84 t^2 x^7 + t^1 x^9 + 9 t^2 x^9$$

ps21 - bps21

0

gs = AllGenerators[8, 1]

$$\{P[\{1\}, 1, 1, 1, 1, 1, 1, 1, 1], xP[\{1\}, 1, 1, 1, 1, 1, 1, 1], x^2P[\{1\}, 1, 1, 1, 1, 1, 1], x^3P[\{1\}, 1, 1, 1, 1, 1], x^4P[\{1\}, 1, 1, 1, 1], x^5P[\{1\}, 1, 1, 1], x^6P[\{1\}, 1, 1], x^7P[\{1\}, 1], x^8P[\{1\}]\}$$

d1 /@ gs

$$\{P[\{2, 1\}, 1, 1, 1, 1, 1, 1, 1, 1] - P[\{2, 1\}, 2, 2, 2, 2, 2, 2, 2, 2], xP[\{2, 1\}, 1, 1, 1, 1, 1, 1, 1, 1] - xP[\{2, 1\}, 2, 2, 2, 2, 2, 2, 2, 2], x^2P[\{2, 1\}, 1, 1, 1, 1, 1, 1, 1, 1] - x^2P[\{2, 1\}, 2, 2, 2, 2, 2, 2, 2, 2], x^3P[\{2, 1\}, 1, 1, 1, 1, 1, 1, 1, 1] - x^3P[\{2, 1\}, 2, 2, 2, 2, 2, 2, 2, 2], x^4P[\{2, 1\}, 1, 1, 1, 1, 1, 1, 1, 1] - x^4P[\{2, 1\}, 2, 2, 2, 2, 2, 2, 2, 2], x^5P[\{2, 1\}, 1, 1, 1, 1, 1, 1, 1, 1] - x^5P[\{2, 1\}, 2, 2, 2, 2, 2, 2, 2, 2], x^6P[\{2, 1\}, 1, 1, 1, 1, 1, 1, 1, 1] - x^6P[\{2, 1\}, 2, 2, 2, 2, 2, 2, 2, 2], x^7P[\{2, 1\}, 1, 1, 1, 1, 1, 1, 1, 1] - x^7P[\{2, 1\}, 2, 2, 2, 2, 2, 2, 2, 2], 0\}$$

d2 /@ (d1 /@ gs)

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

d2ker [8]

```
{x^8 P[{2, 1}], x^8 P[{1, 2}], -x^7 P[{2, 1}, 1] + x^7 P[{2, 1}, 2],
-x^6 P[{2, 1}, 1, 1] + x^6 P[{2, 1}, 2, 2], -x^5 P[{2, 1}, 1, 1, 1] + x^5 P[{2, 1}, 2, 2, 2],
-x^5 P[{2, 1}, 1, 1, 2] + x^5 P[{2, 1}, 1, 2, 2],
-x^4 P[{2, 1}, 1, 1, 1, 1] + x^4 P[{2, 1}, 2, 2, 2, 2],
-x^4 P[{2, 1}, 1, 1, 1, 2] + x^4 P[{2, 1}, 1, 2, 2, 2],
-x^3 P[{2, 1}, 1, 1, 1, 1, 1] + x^3 P[{2, 1}, 2, 2, 2, 2, 2],
-x^3 P[{2, 1}, 1, 1, 1, 1, 2] + x^3 P[{2, 1}, 1, 2, 2, 2, 2],
-x^3 P[{2, 1}, 1, 1, 1, 2, 2] + x^3 P[{2, 1}, 1, 1, 2, 2, 2],
-x^2 P[{2, 1}, 1, 1, 1, 1, 1, 1] + x^2 P[{2, 1}, 2, 2, 2, 2, 2, 2],
-x^2 P[{2, 1}, 1, 1, 1, 1, 1, 2] + x^2 P[{2, 1}, 1, 2, 2, 2, 2, 2],
-x^2 P[{2, 1}, 1, 1, 1, 1, 2, 2] + x^2 P[{2, 1}, 1, 1, 2, 2, 2, 2],
-x P[{2, 1}, 1, 1, 1, 1, 1, 1, 1] + x P[{2, 1}, 2, 2, 2, 2, 2, 2, 2],
-x P[{2, 1}, 1, 1, 1, 1, 1, 1, 2] + x P[{2, 1}, 1, 2, 2, 2, 2, 2, 2],
-x P[{2, 1}, 1, 1, 1, 1, 1, 2, 2] + x P[{2, 1}, 1, 1, 2, 2, 2, 2, 2],
-x P[{2, 1}, 1, 1, 1, 1, 2, 2, 2] + x P[{2, 1}, 1, 1, 1, 2, 2, 2, 2],
-P[{2, 1}, 1, 1, 1, 1, 1, 1, 1, 1] + P[{2, 1}, 2, 2, 2, 2, 2, 2, 2, 2],
-P[{2, 1}, 1, 1, 1, 1, 1, 1, 1, 2] + P[{2, 1}, 1, 2, 2, 2, 2, 2, 2, 2],
-P[{2, 1}, 1, 1, 1, 1, 1, 1, 2, 2] + P[{2, 1}, 1, 1, 2, 2, 2, 2, 2, 2],
-P[{2, 1}, 1, 1, 1, 1, 1, 2, 2, 2] + P[{2, 1}, 1, 1, 1, 2, 2, 2, 2, 2]}
```

d2ker [8] // Length

22