MAT 1300Y (MAT 427S)
TOPOLOGY
P. Selick

½ term
1. Point set topology: Topological spaces and continuous functions, connectedness and compactness, countability and separation axioms.

1½ term.
2. Homotopy: Fundamental group, Van Kampen theorem, Brouwer's theorem for the 2-disk. Homotopy of spaces and maps, higher homotopy groups. Covering theory, universal coverings. categories.
3. Homology: Simplicial and singular homology, homotopy invariance, exact sequences, Mayer-Vietoris, excision, Brouwer's theorem for the *n*-disk, CW-complexes, Euler characteristic. degrees.
4. Cohomology: Cohomology groups, cup products, cohomology with coefficients.
5. Topological manifolds: Orientation, fundamental class, ~~de Rham cohomology for smooth manifolds.~~ ?

Reference:
Hatcher: *Algebraic Topology*, Cambridge University Press, 2001.
Additional References:
Bredon: *Topology and Geometry*, Springer, 1993.
Munkres: *Elements of Algebraic Topology*, Perseus Books, 1993.
Fulton: *Algebraic Topology: A First Course*, Springer, 1994.
Tu Bott: *Differential Forms in Algebraic Topology*, Springer, 1997.

categories? classification of surfaces?
Degree of a map?

Selick: Final though no midterm
HW once a month.

what's an SDR?

About This Class

URL: http://www.math.toronto.edu/~drorbn/classes/0405/Topology/.

Agenda: Learn about the surprising relation between the easily deformed (topology) and the most rigid (algebra).

Instructor: Dror Bar-Natan, drorbn@math.toronto.edu, Sidney Smith 5016G, 416-946-5438. Office hours: Thursdays 12:30-1:30.

Classes: Tuesdays 1-3 and Thursdays 2-3 at Sidney Smith 5017A.

Optimistic Plan:

1. Point set topology: Topological spaces and continuous functions, connectedness and compactness, countability and separation axioms.
2. Homotopy: Fundamental group, Van Kampen theorem, Brouwer's theorem for the 2-disk. Homotopy of spaces and maps, higher homotopy groups.
3. The language of category theory.
4. Covering theory, universal coverings.
5. Homology: Simplicial and singular homology, homotopy invariance, exact sequences, Mayer-Vietoris, excision, Brouwer's theorem for the n-disk, degrees of maps, CW-complexes, Euler characteristic, a word about the classification of surfaces.
6. Cohomology: Cohomology groups, cup products, cohomology with coefficients.
7. Topological manifolds: Orientation, fundamental class, Poincare duality.

Textbooks: We will mainly use James Munkres' Topology (ISBN 0-13-181629-2) and Allen Hatcher's Algebraic Topology (Free! ISBN 0-521-79540-0). Additional texts by Bredon, Bott-Tu, Dugundji, Fulton, Massey and others are also excellent.

Lecture Notes: I'll be happy to scan the lecture notes of one of the students after every class and post them on the web. We need a volunteer with a good handwriting!

The Final Grade: Around the third week of classes (after I'll know you a little better) I will decide on the grading scheme for this class. While the specifics are still open, your final grade is sure to depend on your homework grade and on some final test/report/presentation, and is likely to also depend on a midterm or two.

Problem Sets: There will be about 10 problem sets. I encourage you to discuss the homeworks with other students or even browse the web, so long as you do at least some of the thinking on your own and you write up your own solutions.

Feedback: I'd be very happy to hear from you. There's a link to a feedback form at the top of this class' web site (and here). Anonymous messages are fine, provided they are written with good intent. Though remember that if I don't know who you are I may not be able to address your concern. You will each be *required* to use this feedback form at least once, on the third week of classes (see below).

Class Photo: To help me learn your names, I will take a class photo on Thursday of the third week of classes. I will post the picture on the class' web site and you will be *required* to use the feedback form to identify yourself in the picture.

Math 131 Spring 2003 REALITY
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t's what we really did, as recovered from lecture notes in Sep. 2004:

	M	W	F	Topics
Feb		3	5	Openness and continuity, topology, basis.
	8	10	12	Products, interior&closure, continuity.
	XX	17	19	Infinite products, metric spaces.
	22	24	26	Clopens, connectivity, def. of compactness.
Mar	1	3	5	Compactness, finite products, Lebesgue's lemma, LPC, seq comp.
	8	10	12	Metric compactness, local compactness, AC&Zorn.
	15	17	19	Ultrafilters, Stone-Cech.
	ME	24	26	Hindmann's theorem.
Apr	XX	XX	XX	
	5	7	9	Surfaces and ... classes by Nantel?
	12	14	16	Classes by Nantel?
	19	21	23	Classes by Nantel(?), the fundamental group.
	26	28	30	Van-Kampen
May	3	5	7	Proof of Van Kampen, ...?
	10	12	14	

Math 1300 Topology, Sep 9, 2004

Welcome, go over handout, Lecture notes.

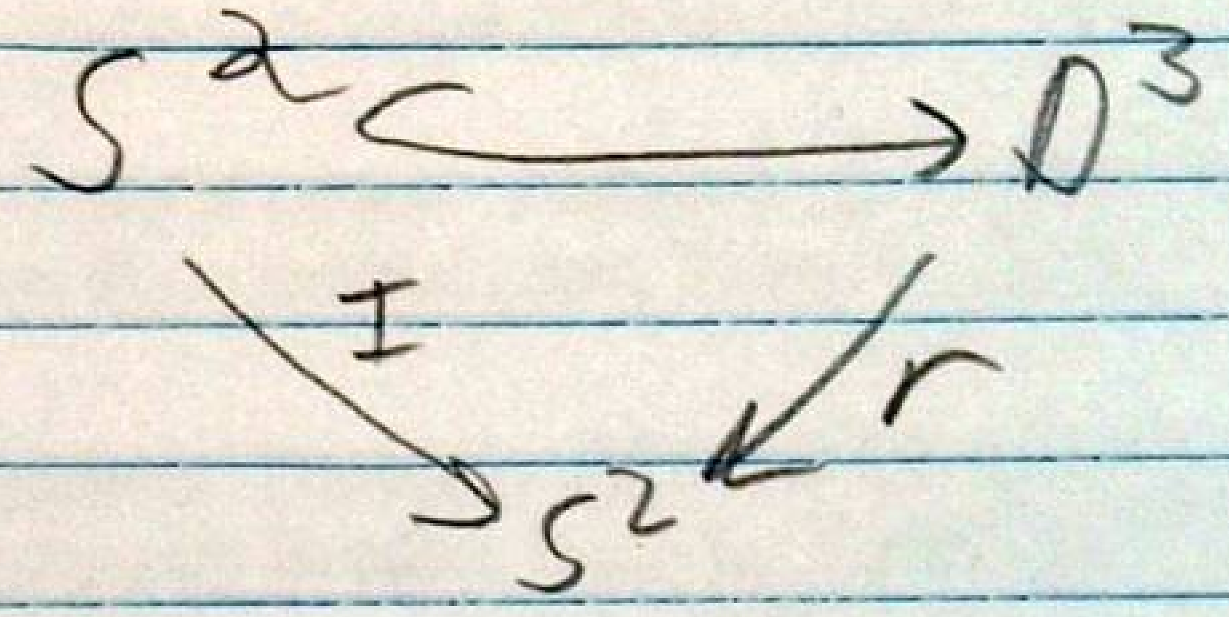
<u>Theorem</u> (The Brouwer fixed point theorem)
Every coninuous map $g: D^3 \longrightarrow D^3$ has a fixed point.

$$D^3 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$$

(What's a cont. map?)

Economic applications!

<u>Lemma</u> Theres no cont. retract $r: D^3 \longrightarrow S^2$



<u>Thm</u> 1. There is a functor H_2 from the category of spaces & maps to the category of groups and homomorphisms.

2. $H_2(S^2) = \mathbb{Z}$ $\quad H_2(D^2) = 0$

1. show mug!
2. class notes.

Math 1300 Topology, Sep 14 2004 (follow March 3, 7, 1996)

Math 1300 Topology, Sep 16 2004

* Pass questionnaire

* Follow March 10, 1996

Math 1300 Topology, Sep 21 2004

* Mandate to go ∈ faster, ∈ tiny. * "About" handout

* Class photo on Thursday. * CCNET?

Def Limit pt. $x \in \overline{A \setminus \{x\}}$ **Thm** $\bar{A} = A \cup A'$

Examples $A = \{\frac{1}{n}\}$; $A = \mathbb{Q}$

2.6 T_2 / Hausdorff spaces
1. $\mathbb{R}$ is ~~Hausd~~ T_2
2. $\mathcal{T} = \{(a, \infty)\}$ isn't

Goodies: In T_2, $x \in A'$ iff every nbd of x intersects A in infinitely many points

→ **Thm** ~~prod~~ product, subspace for T_2

→ **Thm** singletons are closed in T_2

2.7 as on Feb 12, 1993 (cont functions)
1. f^{-1}(open) is open
2. $f(\bar{A}) \subset \overline{f(A)}$
3. f^{-1}(closed) is closed
4. chk on basis
5. cont. at a pt.

$x \in \overline{f^{-1}(A)} \Rightarrow f(x) \in f(\overline{f^{-1}(A)}) \subset \overline{f(f^{-1}(A))} \subset \bar{A} = A$.

2.8 as on Feb 17, 1993

+
1. X_α triv $\forall \alpha \Rightarrow \prod X_\alpha$ triv
2. X_α T_2 $\forall \alpha \Rightarrow \prod X_\alpha$ T_2
3. X_α discrete $\forall \alpha$ ~~⇒~~ $\prod X_\alpha$ discrete.

2.9 Metric spaces, balls,
Basis to open sets.
Metric top is always T_2

Thm $\prod_{i \in \mathbb{N}} X_i$ is metrizable iff each X_i is.

Homework Assignment 1

Assigned Tuesday September 28; due Thursday October 14, 3PM, in class

Required email. The class photo will be on the class' web site in a day or two and you are all required to find it, find yourself in the photo, and send me an email message (either using the feedback form on the class' web site or using my regular email address) with the following information:

- Where are you in the picture? (Say something like "back row 3rd from the left", and to be sure, add something descriptive like "I'm the one with the knotted hair and the Möbius band tattoo on my forehead".)
- Your name.
- Your email address.
- Your telephone number (optional).
- Which of the last four pieces of information do you allow me to put on the web? If you don't write anything about this, I'll assume that your location in the photo, your name and your email address are public but that your phone number is to be kept confidential.

Your email is due earlier than the rest of this assignment, on Monday October 4 at 4PM. If you aren't in the picture at all, find me before Monday and I'll take a (small) picture of you on the spot and edit it into the main picture.

Required reading. Read, reread and rereread your notes to this point, and make sure that you really, really really, really really really understand everything in them. Do the same every week! Also, read all of Munkres chapter 2.

Solve the following problems. (But submit only the underlined ones). In Munkres' book (Topology, 2nd edition), problems 4, <u>8</u> on pages 83–84, problems <u>4</u>, 8 on page 92, problems 6, 7, <u>13</u> on page 101, problems 9, 11, 12, <u>13</u> on page 112, problems <u>6</u>, 7 on page 118 and problems <u>3</u>, 8 on pages 126–128. Also solve (but don't submit) the following

Problem. Let C be the "Cantor set", the closure of the set of real numbers in $[0,1]$ whose expansion to base 3 doesn't contain the digit 1 (e.g., $\frac{3}{4} = 0.2020202020202\ldots$ is in, but $\frac{1}{2} = 0.11111111\ldots$ is out). Prove that C (taken with the topology induced from $\mathbb{R}$) is homeomorphic to $\{0,1\}^{\mathbb{N}}$ (taken with the product topology).

Agenda for September 28, 2004

Comment. f is continuous at x iff for every neighborhood V of $f(x)$, its inverse image $f^{-1}(V)$ *contains* a neighborhood of x.

Agenda. We will discuss two primary notions and the interaction between them and along the way also learn about sequences....

First notion — the product topology. (The naive definition and the box topology), definition by listing our requirements, uniqueness and existence, interaction with the trivial topology, the subspace topology, T_2 and the discrete topology.

Second notion — metric spaces and metrizability Definition, examples, the metric topology, T_2-ness, metrizability.

The interaction We'll prove three theorems:

Theorem 1. (good) $\emptyset \neq \prod_{k=1}^{\infty} X_k$ is metrizable iff every X_k is metrizable.

Theorem 2. (who cares?) $\mathbb{R}^{\mathbb{N}}_{\text{box}}$ is not metrizable.

Theorem 3. (bad) $\mathbb{R}^{\mathbb{R}}$ is not metrizable.

In order to prove Theorems 2 and 3 we will need to know about sequences, and these are quite interested by themselves:

Sequences. Convergence, sequential closure.

Proposition 1. The sequential closure is always a subset of the closure, and in a metrizable space, they are equal.

Proposition 2. If $F : X \to Y$ and X is metric, then f is continuous iff for every sequence in X, the convergence $x_k \to x$ implies the convergence $f(x_k) \to f(x)$.

Mon Nov 1 6-8 Makeup.

Sep 23 was cancelled.

* Finish 5 minutes early for class photo

* Makeup on wk of Nov 1st.

Math 1300 Topology, Sep 28 2004

* Discuss H/W / Finish 5 min early

* Set makeup time

* Discuss agenda.

on board topologies on $\prod X_\alpha$

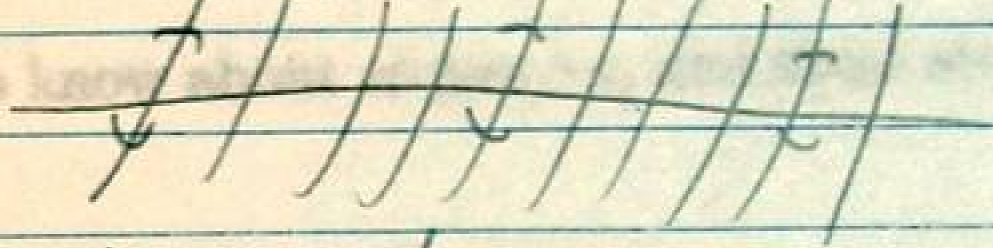
box – generalize basis | product/Tychonoff – generalize requirements

basis $\prod U_\alpha$

1. π_α cont. 2. If f_α cont. $\Rightarrow \prod f_\alpha$ cont.

Thm (exists & unique)

basis $\pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \ldots \cap \pi_{\alpha_n}^{-1}(U_{\alpha_n})$



"cylinders"

Triv., T_2, ~~discrete~~, subspace, discrete $\emptyset$

Def Metric, ball, baseness.

Def Metrizable Example if d is a metric, so is $\bar{d} := \min(d, 1)$ & they define the same topology.

Examples $\mathbb{R}^n$, discrete, $C_0(X, \mathbb{R})$

Pf of thm 1

convergence of a seq, seq closure,

Prop 1

Pf of thm 2 & 3

Pf of Prop 2

If time – $C_0(X, Y)$ is closed in $C_0(X_{discrete}, Y)$.

*class photo

Math 1300 Topology, Sep 30, 2004

Finish sep 28 material

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clopen sets, connectedness.

<u>Thm</u> $X \subset \mathbb{R}$ is connected iff it is an interval or a ray.

Math 1300 Topology, Oct 5 2004

The above theorem and continue as on March 28, 1996

Math 1300 Topology, Oct 7 2004 1hr

Compactness as on April 21, 1996

Math 1300 Topology, Oct 12 2004 2hrs

1. Compactness via the finite intersection property

~~Example~~ every countable subset of a compact set has a limit point.

2. Tychonoff as on April 28, 1996

3. Zorn'lemma & the fishy AC

4. AC to good to be true. $\exists$ Lim

5 $\beta\mathbb{N} := \overline{\iota(\mathbb{N})}$

$$\iota : \mathbb{N} \longrightarrow \prod_{\substack{f:\mathbb{N}\to\mathbb{R} \\ \text{bdd}}} \overline{[\mathrm{ran}(f)]}$$

Math 1300 Topology, Oct 14 2004 1 hour

Redefine $\beta\mathbb{N}$ and prove the existence of Lim.

Continue as on May 2, 1996

$$\mathrm{Lim} : \left\{ \begin{matrix} \text{bndd} \\ \text{sequences} \end{matrix} \; a : \mathbb{N} \to \mathbb{R} \right\} \longrightarrow \mathbb{R}$$

s.t.
1. $\mathrm{Lim}(a_n + b_n) = \mathrm{Lim}(a_n) + \mathrm{Lim}(b_n)$
2. $\mathrm{Lim}(a_n b_n) = \mathrm{Lim}(a_n) \cdot \mathrm{Lim}(b_n)$
3. $\mathrm{Lim}(f(a_n)) \underline{=f}(\mathrm{Lim}(a_n))$ (f cont.)
4. $\mathrm{Lim}\, a_n \in \overline{\{a_n\}}$; $\mathrm{Lim}(a_1, a_2, \ldots a_n, 00000) = 0$
5. ~~$\mathrm{Lim}\, a_{n+1}$~~ $= \mathrm{Lim}\, a_n$

$$\beta\mathbb{N} := \overline{\iota(\mathbb{N})} \subset \prod_{\{a\}} \overline{\mathrm{Ran}(a)} \qquad \iota(n)_a = a_n$$

$$\mu \in \beta\mathbb{N} \setminus \mathbb{N} \iff \mathrm{Lim}(a_n) = \mu_a$$

Lims & ultra filters.

מבוא לטופולוגיה, 2 במאי 1996

<u>משפט</u> (גרסה 2 הלמה של אוריסון) אם X קומפקטי ו- $x\neq y\in X$ אזי יש $f:X\to I$ רציפה עם $f(x)=0$; $f(y)=1$

יוכח בהמשך. רגולרי, ואפשר להפריד את x מקבוצה סגורה A עם $y\notin A$

נורמלי, ואפשר להפריד את y מקבוצה סגורה B עם $A\cap B=\emptyset$.

זה משפט <u>אוריסון</u> חזק.

<u>משפט</u> כל מ"ט קומפקטי X ניתן לשיכון בקוביה $I^{\mathcal{F}_X}$

<u>הגדרה</u> $\mathcal{F}_X=\{f:X\to I \text{ רציפה}\}$, ע"י $\tau_X: X\to I^{\mathcal{F}_X}$

$$(\tau_X(x))_f = f(x)$$

צ"ל רציפות, סגירות, חח"ע.

<u>הגדרה</u> X רגולרי לחלוטין אם לכל $A\subset X$ סגורה ולכל $y\notin A$ יש רציפה $f:X\to I$ עם $f(A)=0$, $f(y)=1$

<u>טענה</u> מרחב X הוא רגולרי לחלוטין אמ"מ $\{[f\neq 0] : f\in\mathcal{F}_X\}$ בסיס לטופולוגיה.

<u>משפט</u> אם X רגולרי לחלוטין, אז $\tau_X: X\to I^{\mathcal{F}_X}$ שיכון.

<u>מסקנה</u> לכל מ"ט רגולרי לחלוטין X יש מ"ט קומפקטי המכיל אותו: $\beta X=\overline{\tau_X(X)}$

<u>משפט</u> βX הוא קומפקטיפיקציית סטון-צ'ך של X.

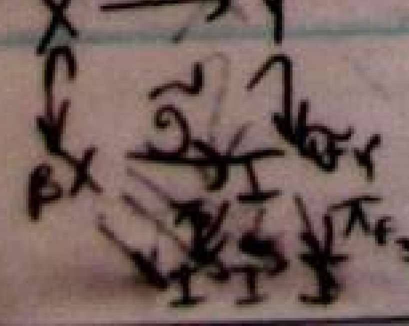
צ"ל 1. βX קומפקטי האוסדורף ומכיל את X

2. כל פונקציה רציפה $g:X\to Y$ (קומפקטי האוסדורף) ניתנת להרחבה יחידה $\hat{g}:\beta X\to Y$

א. יחידות: נובעת כי X צפופה ב- βX

ב. אם $g:X\to I$: $\beta X\subset X^{\mathcal{F}_X}\xrightarrow{\pi_g} I$

ג. $g:X\to Y$



$$\hat{g}(\beta X)=\hat{g}(\overline{X})\subset\overline{g(X)}\subset\overline{Y}=Y$$

מבוא לטופולוגיה, 5 במאי 1996.

<u>משפט</u> (גרסה של אוריסון) X קומפקטי, A,B סגורות זרות ⇐ יש רציפה $f:X\to I$ עם $f(A)=0$ ו- $f(B)=1$.

<u>הגדרה</u>: $\mathcal{F}_X=\{f:X\to I$ רציפה$\}$; $\iota:X\to I^{\mathcal{F}_X}$ ע"י $(\iota(x))_f=f(x)$

נזכר ש-ι שיכון את X ב"קובייה" לטיכונוף.

<u>הגדרה</u> $\beta X=\overline{\iota(X)}$ (מ"ט קומפקטי שמכיל את X)

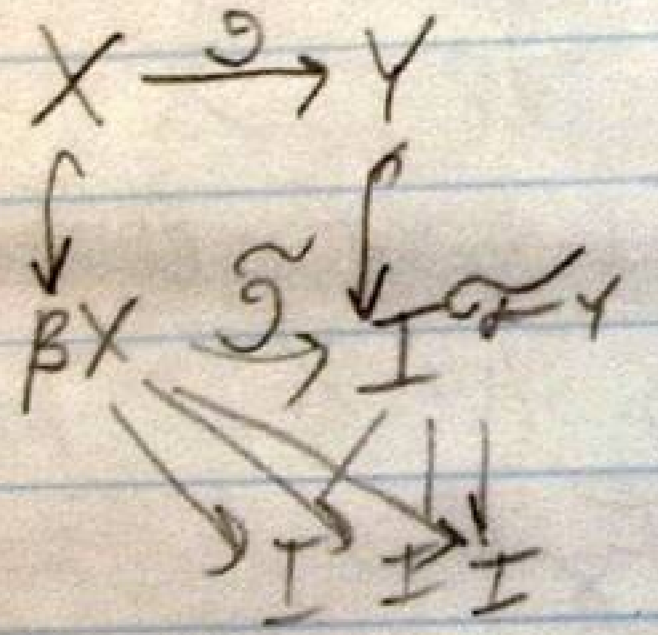
<u>משפט</u> βX הוא קומפ' סטון-צ'ך של X, כלומר

1. βX קומפקטי האוסדורף ומכיל את X.
2. כל רציפה $g:X\to Y$ (Y קומפקטי) ניתנת להרחבה יחידה $\tilde g:\beta X\to Y$

א. יחידות כי X צפופה ב- βX.

ב. אם $g:X\to I$ אז $X\hookrightarrow\beta X\xrightarrow{\iota} I^{\mathcal{F}_X}\xrightarrow{\pi_g} I$

ג. $g:X\to Y$



$\tilde g(\beta X)=\tilde g(\overline{X})\subset\overline{g(X)}\subset\overline{Y}\subset Y$

אולי היה כדאי לבצע את שלב זה אחרת.

<u>משפט</u> קומפ' סטון-צ'ך של מ"ט רגולרי לחלוטין קיים ויחיד.

<u>בעיות</u>: 1. Lim מכליל את lim

2. הלמה של אוריסון:

<u>הגדרה</u> מ"ט X נקרא נורמלי אם $A,B\subset X$ סגורות זרות ⇐ קיימות U,V פתוחות זרות עם $A\subset U, B\subset V$

<u>משפט</u> 1. כל מרחב מטרי הוא נורמלי.
2. כל מ"ט קומפקטי הוא נורמלי.

הלמה של אוריסון: אם X נורמלי ו- $A,B\subset X$ סגורות זרות, אזי יש $f:X\to I$ רציפה עם $f(A)=0$, $f(B)=1$.

<u>הוכחה</u>: א. לכל רציונלי $0<q\leq 1$ מוצאים פתוחה U_q עם $A\subset U_q\subset B^c$ וגם $q<r\Rightarrow \overline{U_q}\subset U_r$

ב. מרחיבים $U_{<0}=\phi$ ו- $U_{>1}=X$ ומגדירים $f(x)=\inf\{q: x\in U_q\}$

ג. f מקיימת את הדרוש.

Math 1300 Topology, Oct 19 2004

Today's goal: Soft landing

On board: $\beta N := \overline{\iota(N)}$

$\iota : N \to \prod_{\text{bdd } (a_n)} [\,\inf a_n, \sup a_n]$

soft landing 1. More on βN
2. What else has β
3. Sep. axioms

* More properties of βN:

1. Size
2. pts are ultrafilters. (0–1 finitely additive measures on N)
3. Addition: Associative, not-commutative, cont. on one side } quickly

Leads to a proof of Hindman's theorem!

Definition compactification

Stone Čech compactification.

Thm If βX exists, it is unique.

Thm βX exists iff X is "completely regular", $T3\frac{1}{2}$

Def completely regular: Topology can be seen by real functions: $\{[f \neq 0] : f \in \mathcal{C}(X)\}$ basis for top.

iff A closed, $x \in A^c \Rightarrow$ can be sep.

Thm (hard) compact spaces are completely regular.

Claim A subspace of $T3\frac{1}{2}$ is $T3\frac{1}{2}$

$$\iota : X \to I^{\mathcal{C}_I(X)} \qquad \beta X := \overline{\iota(X)}$$

this works as on May 5, 2004

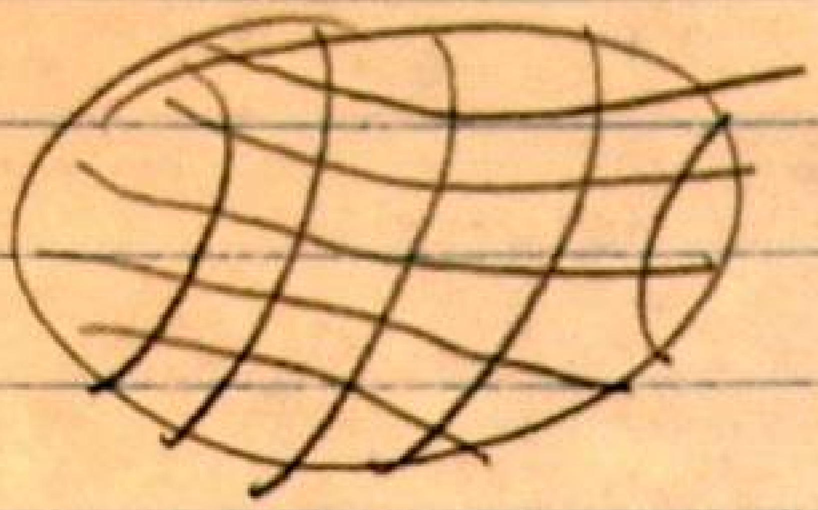
cont...

Def Normal (T_4)

Thm Metric ∨ compact $\Rightarrow T_4$

Thm (Urysohn's Lemma)

Math 1300 Topology, Oct 21 2004



coordinate systems & embeddings into cubes

$$\iota : X \longrightarrow I^{C_I(X)}$$

If X $T_{3\frac{1}{2}}$ — embedding

If X compact — it is a closed subspace of a cube

$$\beta X = \text{"}\overline{X}\text{"} := \overline{\iota(X)}$$

<u>Thm</u> if $f : X \longrightarrow Y$ cont. & Y compact,

$\exists !\ \tilde{f} : \beta X \longrightarrow Y$ s.t. $\tilde{f}|_X = f$.

<u>Def</u> Normal (T_4)

<u>Thm</u> Metric $\vee$ compact $\Longrightarrow T_4$

<u>Thm</u> Urysohn's Lemma as on May 9, 1996

Point set topology:

1. Urysohn
2. Tietze
3. compact metric spaces
4. Baire.

No class Thursday, Yes class Mon 6-8.

Math 1300 Topology, Oct 26 2004

Urysohn & Tietze as on May 9, 1996

Compactness in metric spaces as on May 12, 1996

Math 131, April 21 1993

Nantel's material: triangulations, disks w/ identifications, $T^2 \# \mathbb{RP}^2 \cong \#^3 \mathbb{RP}^2$; eliminate aa^{-1}; one vertex; No ..b...b.. ; No ...a...a^{-1}....

Euler char, invariance,

$\chi(\#^n T^2)$? $\chi(\#^n \mathbb{RP}^2)$?

Groupness of π_1; pushforwards φ_*; functoriality; homotopy of maps; retracts; def ret; simply conn; contractible; $\pi_1(S^1)$.

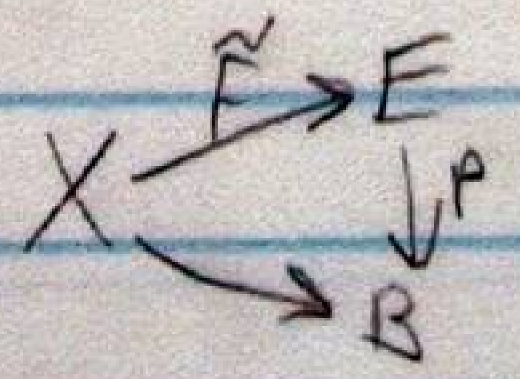
<u>Brouwer fixed pt.</u>: 1. state 2. why a. eqns b. flows
3. No $\Rightarrow$ retract
4. $S^1 \xrightarrow{i} B^2 \xrightarrow{r} S^1$

<u>Def</u> $p: E \to B$ cont onto
U (open) evenly covered $\Leftrightarrow p^{-1}(U) = \bigcup V_\alpha$, V_α open, $p|_{V_\alpha}$ is homeo
V_α's are "slices".

<u>Def</u> Covering map $p: E \to B$ E "covering space".

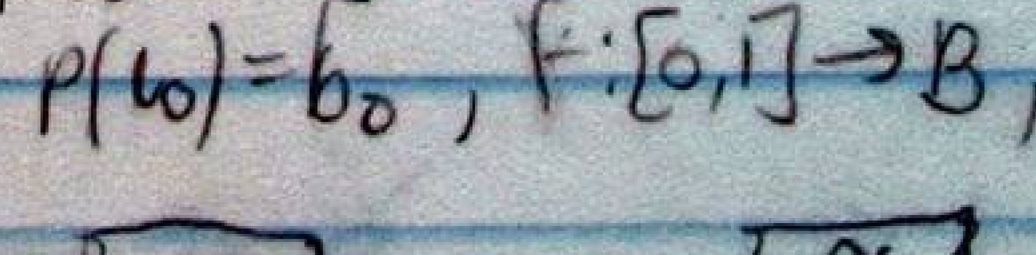
<u>Eg</u>: $\mathbb{R} \to S^1$, $S^1 \xrightarrow{n} S^1$

<u>Def</u> lifting



The path lifting prop: $p: E \to B$ covering map
$p(e_0) = b_0$, $f: [0,1] \to B$, $f(0) = b_0 \Rightarrow \exists! \tilde{f}$

The homotopy lifting prop



<u>Thm</u> $\pi_1(S^1) = \mathbb{Z}$

HW

*candies
*Term Exam
*Classnotes
*HW1
*Harvard handouts.

Math 1300 Topology, Nov 1 2004 (Makeup class)

on board

TFAE for a metric space X:

1. X is compact.
2. X is limit point compact.
3. X is sequentialy compact
4. X is "totally bounded" & satisfies Lebesgue's Lemma
5. X is totally bounded & complete.

Pf: 1⇒2, 2⇒3 done

Define "totally bounded"

State "Lebesgue Lemma"

Aside & pf: if X is metric compact & f: X → ℝ is continuous, then f is uniformly cont. (where is the #/use)

Pf of 3⇒4: 1. X is totally bounded (by contradiction)
2. Lemma: Every cont. function on X attains its max.
Pf of Lemma
Thm from Lemma. $f(x) = \sup\{\delta : \exists U \in \mathcal{U}, B_\delta(x) \subset U\}$

Pf of 4⇒1 Easy

Define "complete" (Every Cauchy seq. converges)

Pf of 3⇒5

Pf of 5⇒3

Aside: Every metric space has a "completion" – a complete metric space in which X is isometrically embedded densely.

Definition Baire space: "The union of countably many nowhere-dense sets has no interior." or "the intersection of countably many open dense sets is dense."

Theorem Complete metric spaces are Baire.
Compact T_2

Theorem "Most" continuous functions are nowhere diff'able.

Math 1300 Topology, Nov 1 2004 Makeup. (Monday)

Finish TFAE for compact metric.

Math 1300 Topology, Nov 2 2004

<u>comment 1</u> reprove tot bndd + complete ⇒ seq compact by "taking commitments"

<u>comment 2</u> Every metric space has a unique "completion" - a complete metric space in which it is densely embedded.

<u>Def</u> A Baire space: A countable union of thin sets is nowhere dense ⟺ A countable intersection of thick sets is dense.

<u>Thm</u> metric & complete / ~~compact~~ / locally compact ⇒ Baire

<u>Example</u> $X = (\mathcal{C}([0,1]), \| \|_\infty)$ $\quad U_n = \left\{ f : \begin{array}{l} \exists \text{ partition } 0 = x_0 < x_1 \ldots < x_{g0} \\ \forall i \ |x_{i+1} - x_i| < \frac{1}{n} \\ \& \ \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} > n \end{array} \right\}$

<u>claim</u> U_n is open dense

<u>claim</u> $g \in \bigcap U_n \Rightarrow g$ is nowhere diff-able.

<u>Example</u> The uniform boundedness principle.

---

pf of theorem

The fundamental group as on June 9, 1996.

*Exam: ~~Nov~~ Tue Nov 16 6-8PM

* HW 2.

Math 1300 Topology, Nov 4 2004

on board:

Baire space: the union of countably many thin sets has no interior, the intersection of countably many thick sets is dense.

thin: closure has no interior

thick: interior is dense

Thm complete metric ⟹ Baire

Thm compact ⟹ Baire

Def locally compact.

Def αX claim it is compact.

claim A space is loc. compact iff it is a compact space with one pt. removed.

The fundamental group as on June 9, 1996.

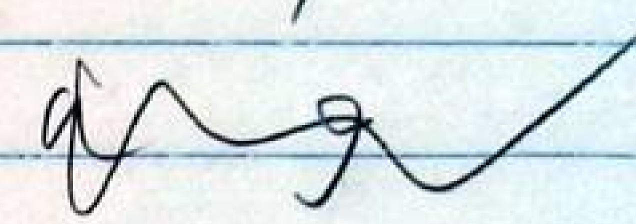
Math 1300 Topology, Nov 9 2004

Discuss TE1

- * Tue Nov 16 6-8 PM
- * Material
  - * All that was said in class
  - * All HW.
- * Exam:
  - * HW repeated
  - * Class proofs.
  - * Fresh exercises.

A based space (X, b)

Def. A path γ, homotopy H, equiv relation

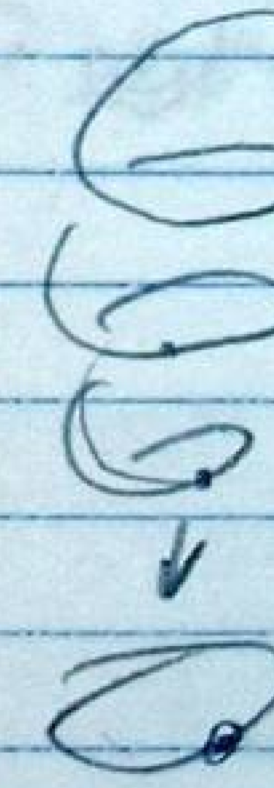
The product law, 

$\pi_1(X, b)$ is a group.

Examples $\pi_1(\mathbb{R}^n) = 0 = \{1\}$

Thm 1 $\pi_1(S^1) = \mathbb{Z}$

Def X covers B



"the path-lifting property"

First formulate for 1 pt.

Thm 2 Given $F : Y \times [0,1] \longrightarrow B$ and $\tilde{F}_0 : Y \longrightarrow X$ lifting $F_0 : Y \to X$ $(F_0 = F|_{Y \times \{0\}})$,
$\exists! \tilde{F} : Y \times [0,1] \longrightarrow B$ extending $\tilde{F}_0$ & lifting F.

Pf of 1 from 2

Pf of 2

TE1: Tue Nov 11 6-8 PM
SS 1087

Math 1300 Topology, Nov 11 2004

Cont. as on Nov 9, 2004.

Web last handout!

Math 1300 Topology, Nov 16 2004.

1. Finish path lifting 2. Functoriality of π_1

3. Homotopy invariance of π_1 (under maps, under homotopy equiv.)

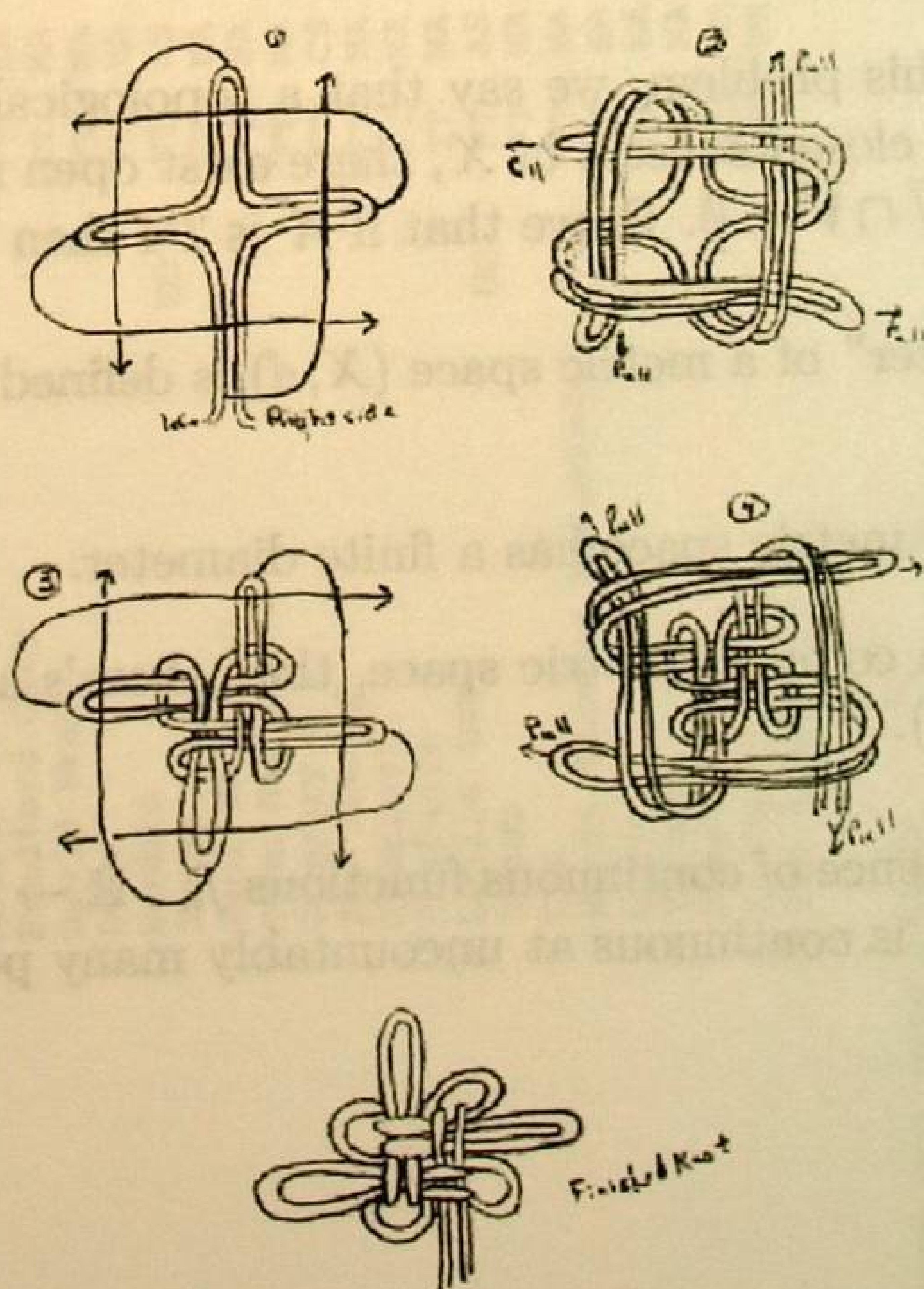
4 Brouwer.

Math 1300Y Topology — Term Exam 1

University of Toronto, November 16, 2004

Solve 5 of the following 6 problems. Each problem is worth 20 points. If you solve more than 5 problems indicate very clearly which ones you want graded; otherwise a random one will be left out at grading and it may be your best one! You have an hour and 50 minutes. No outside material other than stationary is allowed.

Good Luck Knot



(from http://www.mresource.com/Fiber/COEPart2/goodluckknot.htm)

Problem 1. Let X be an arbitrary topological space. Show that the diagonal $\Delta = \{(x,x) : x \in X\}$, taken with the topology induced from $X \times X$, is homeomorphic to X. (18 points for any correct solution. 20 points for a correct solution that does not mention the words "inverse image", "open set", "closed set" and/or "neighborhood".)

Problem 2. Let (X,d) be a connected metric space and let x and y be two different points of X.

1. Prove that if $0 \le r \le d(x,y)$ then the sphere of radius r around x, $S_r(x) := \{z : d(x,z) = r\}$, is non-empty.
2. Prove that the cardinality of X is at least as big as the continuum: $|X| \ge 2^{\aleph_0}$.

Problem 3.

1. Define "X is completely regular".
2. Prove that a topological space X can be embedded in a cube (a space of the form I^A, for some A) iff it is completely regular.

Problem 4.

1. Define "X is $T4$ (normal)".
2. For the purpose of this problem, we say that a topological space is $T4\frac{1}{4}$ if whenever A and B are disjoint closed subsets of X, there exist open sets U and V in X so that $A \subset U$, $B \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$. Prove that if X is $T4$ then it is also $T4\frac{1}{4}$.

Problem 5. The "diameter" of a metric space (X,d) is defined to be $D_X := \sup\{d(x,y) : x,y \in X\}$.

1. Prove that a compact metric space has a finite diameter.
2. Prove that if X is a a compact metric space, then there's a pair of points $x_0, y_0 \in X$ so that $D_X = d(x_0, y_0)$.

Problem 6. If f_n is a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n(x) \to f(x)$ for each $x \in \mathbb{R}$, show that f is continuous at uncountably many points of $\mathbb{R}$.

Good Luck!

Math 1300 Topology, Thursday Nov 18 2004.

The story with Q6. (See Munkres Thm 48.5)

Homotopy theory light

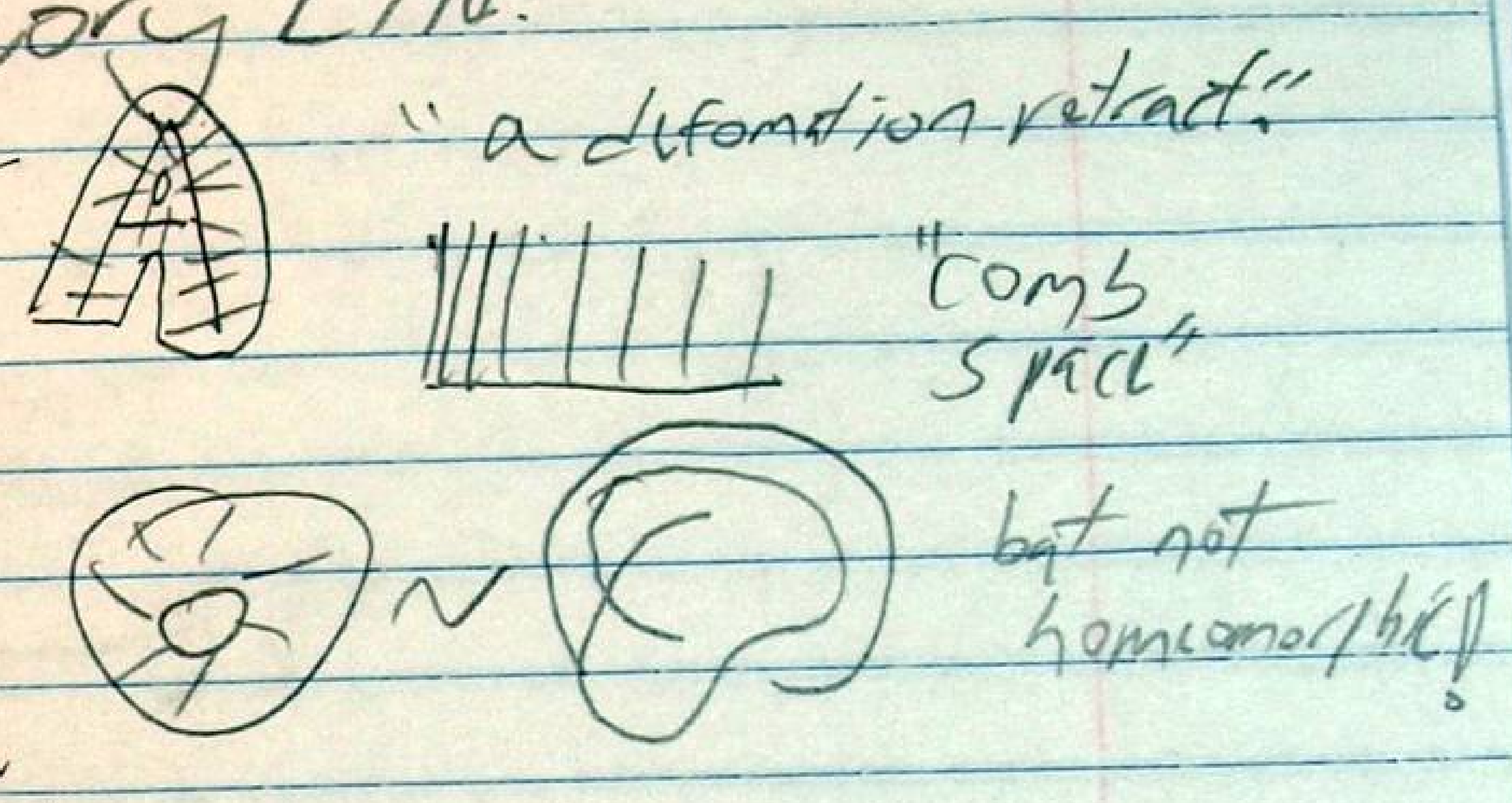
* Product spaces
* change of basepoint.
* Functoriality.
* Homotopic maps.
* Homotopy equivalences.
* No retract $r : D^2 \to S^1$
* Brouwer in D^2.
* The fundamental theorem of algebra.

Math 1300 Topology, Tues Nov 23 2004

* TEI apology again.
* Distribute TEI & HW3
* Gödel's theorem?
* Groups defined by generators and relations?

Continue with Homotopy Theory Lite:

* functoriality
* Homotopic maps
* Homotopy equivalences?



Fun

* No retract $r: D^2 \to S^1$
* Brouwer in D^2 $\quad \mathbb{R}^2 \not\cong \mathbb{R}^n$ $n \neq 2$
* The fundamental theorem of algebra.
* Borsuk-Ulam (So S^2 isn't a subspace of $\mathbb{R}^2$)

(If $S^2 = A_1 \cup A_2 \cup A_3$ (closed sets) then at least one of them contains antipodals) (ask about 4)

pushouts?

*Grading Issues

* $\pi_1(S^2)$

Math 1300 Topology, Thursday Nov 25 2004

If $\gamma: S^1 \to S^1$ is even, deg γ is even
is odd deg γ is odd

Pf 1: lift $[0, 1/2]$

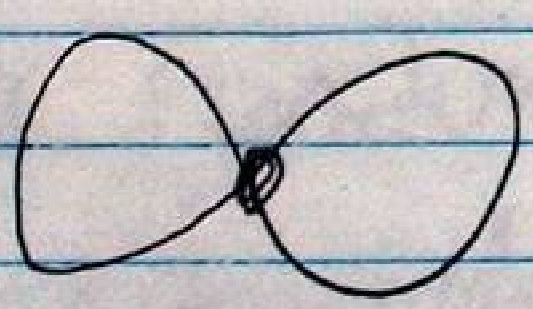
Pf 2 Use ~~group~~ Topological groups

Borsuk-Ulam 1. S^2 is not a subspace of $\mathbb{R}^2$

2. $S^2 = A_1 \cup A_2 \cup A_3$ closed sets, at least one contains a pair of antipodes. (ask about 4)

Tues Nov 30 1200:
Van-Kampen: under Favorable conditions

$$\pi_1(U \cup V) = \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

Examples 1. 

2. T^2

3. $\mathbb{P}^2$

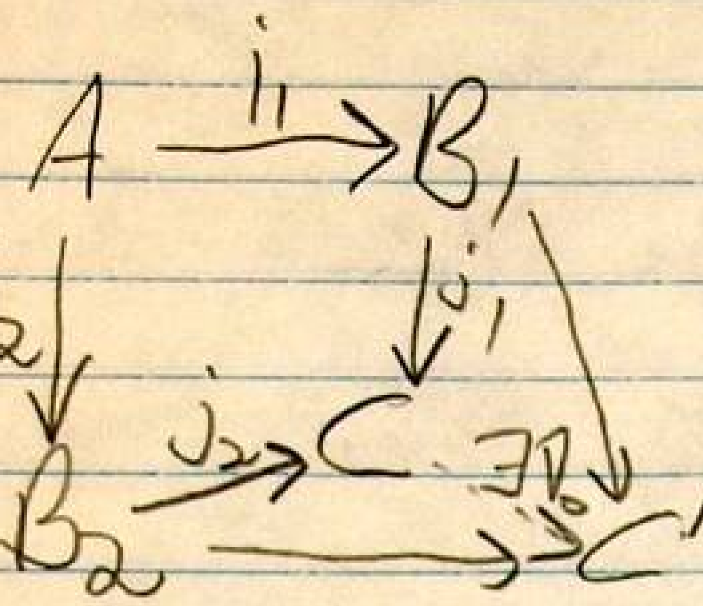
4. S^n $n \geq 2$

5. $(T(p,q))^c$

Any finitely presented group is $\pi_1(X)$!

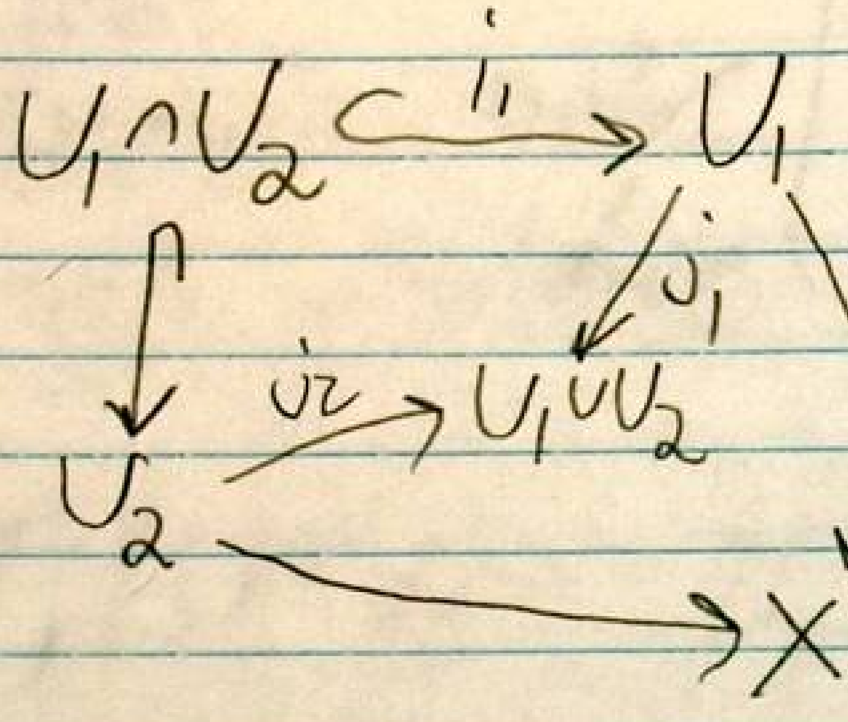
Math 1300 Topology, Thursday Dec 2, 2004

* Push outs in a general $\mathcal{C}$

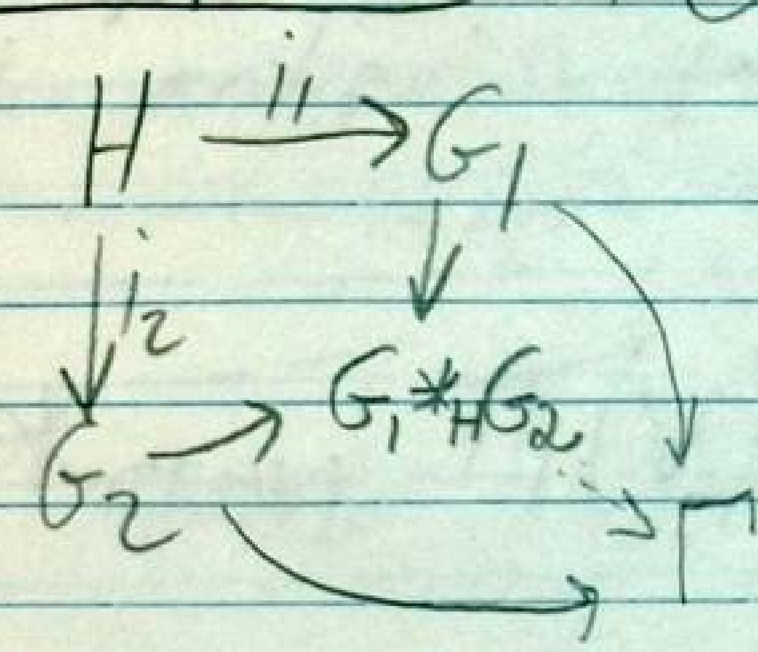


claim C is unique up to isomorphism.

Example 1 in Top: (First in Set)



Example 2 in Groups:



Van Kampen if U_1, U_2 are open (in $U_1 \cup U_2 = X$) and $b \in U_1 \cap U_2$ and $U_1 \cap U_2$ is pathwise connected, the

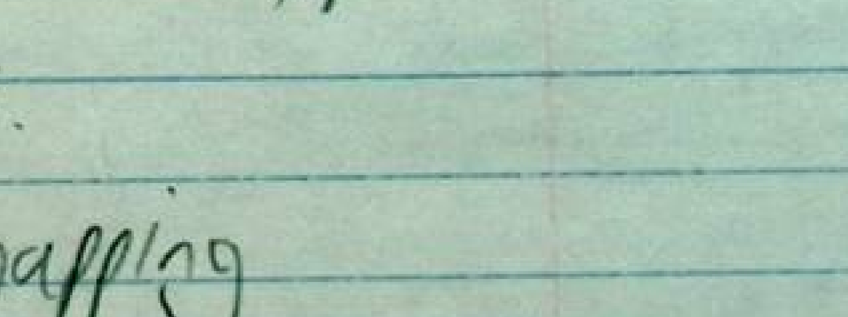
$$\pi_1(U_1 \cup U_2) = \pi_1(U_1) *_{\pi_1(U_1 \cap U_2)} \pi_1(U_2)$$

$$= \pi_1(U_1) * \pi_1(U_2) \Big/ {\scriptstyle \forall \gamma \in \pi_1(U_1 \cap U_2)} \; i_1\gamma = i_2\gamma$$

Pf $\alpha : G_1 *_H G_2 \longrightarrow \pi_1(U_1 \cup U_2)$ — obvious.

$\beta : \pi_1(U_1 \cup U_2) \longrightarrow G_1 *_H G_2$ by mapping

$\gamma \longmapsto \gamma_\rho^b := \gamma_1 \rho_1^{-1} \cdot \rho_1 \gamma_2 \rho_2^{-1} \cdots$

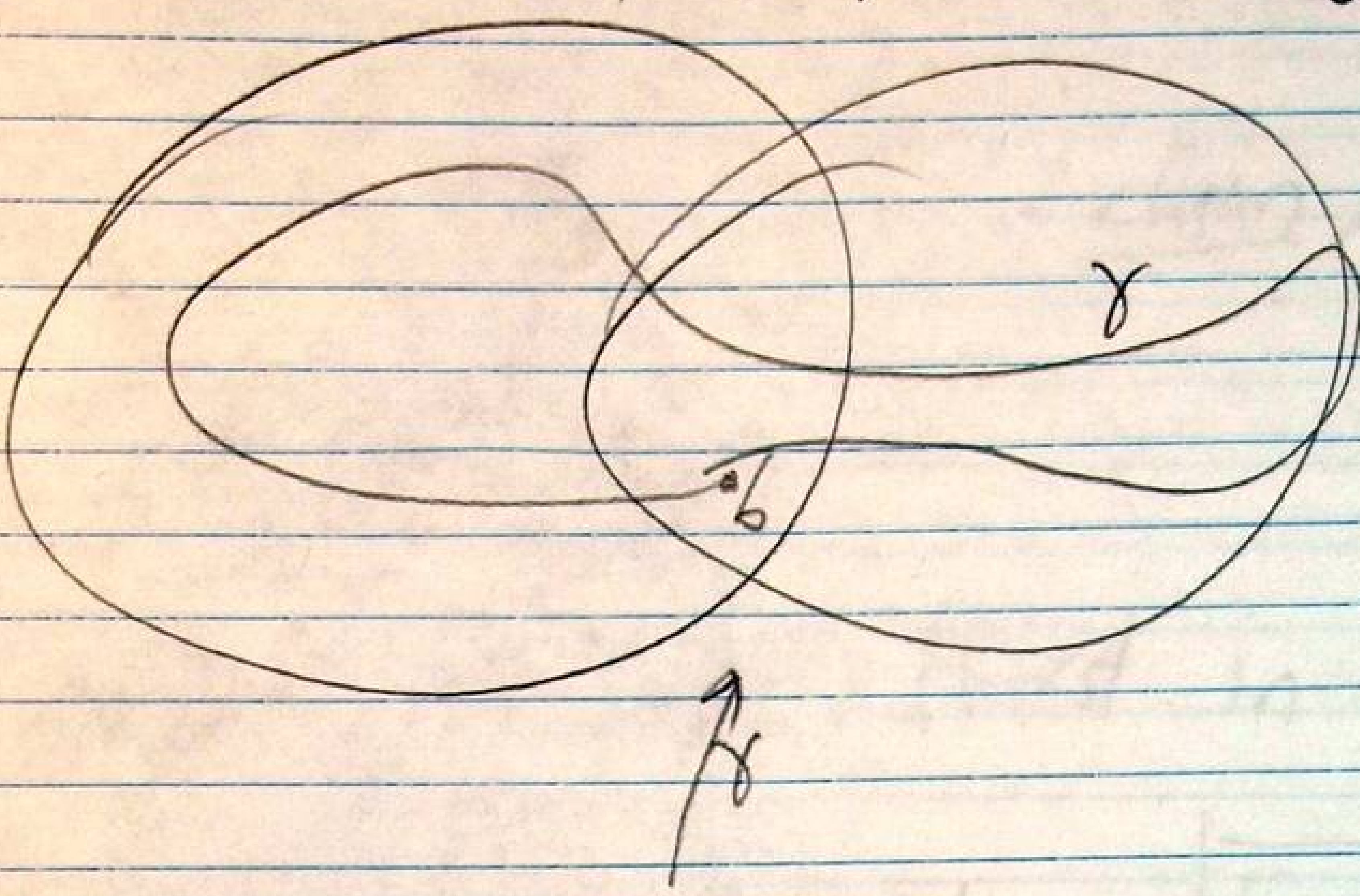


where γ^b is γ with $\gamma(t_i)$ pulled to b along ρ_i

1. dep on b.
2. dep on ρ.
3. dep on γ.

Math 1300 Topology, Tuesday Dec 7 2004

Goal: Prove Van-Kampen and go.

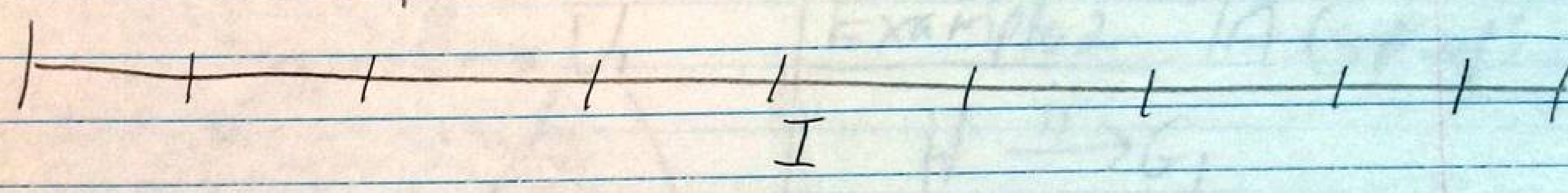


$G_1 = \pi_1(U_1)$

$G_2 = \pi_1(U_2)$

$H = \pi_1(U_1 \cap U_2)$

$\eta : G_1 *_H G_2 \to \pi_1(U_1 \cup U_2)$ the obvious.



Define $\sigma : \pi_1(U_1 \cup U_2) \to G_1 *_H G_2$ in steps.
($\gamma \in \pi_1(U_1 \cup U_2)$)

1. Partition $[0,1]$ so that $\gamma(\text{each part}) \subset U_1$ or U_2
2. Consolidate so $\gamma(t_i) \in U_1 \cap U_2$ & write $\gamma = \gamma_1 \gamma_2 \gamma_3 \ldots$
3. Pinch at each t_i choosing β_i w/ $\beta_i(0) = \gamma(t_i)$, $\beta_i(1) = b$ in $U_1 \cap U_2$
4. Set

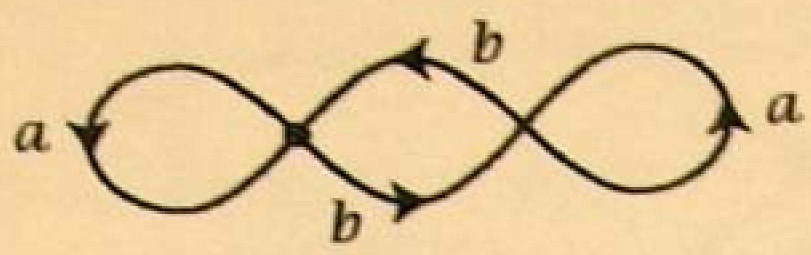
$$\sigma_{\beta_1 \beta_i}(\gamma) = (\gamma_1 \beta_1)(\beta_1^{-1} \gamma_2 \beta_2)(\beta_2^{-1} \gamma_2 \beta_3) \ldots$$

<u>Claim</u> 1. Invariant under subdivision

2. Indep. ~~wds~~ of β_i
3. homotopy invariant.
4. $\sigma \circ \eta = I_{G_1 *_H G_2}$
5. $\eta \circ \sigma = I_{\pi_1(U_1 \cup U_2)}$

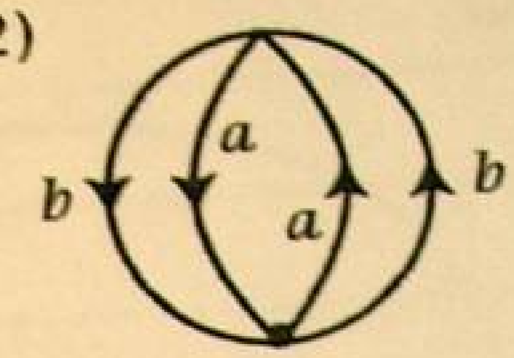
Some Covering Spaces of $S^1 \vee S^1$

(1)



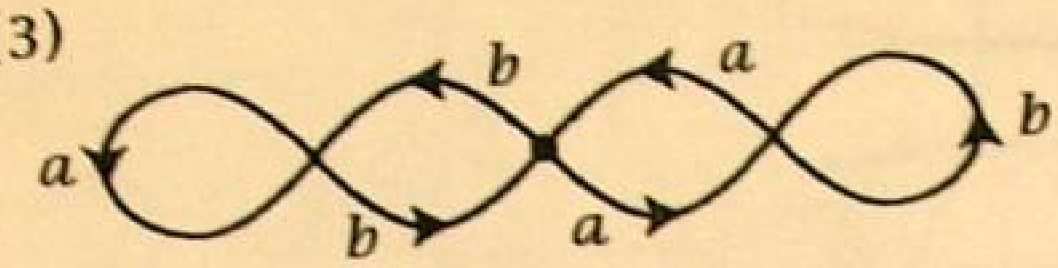
$\langle a, b^2, bab^{-1} \rangle$

(2)



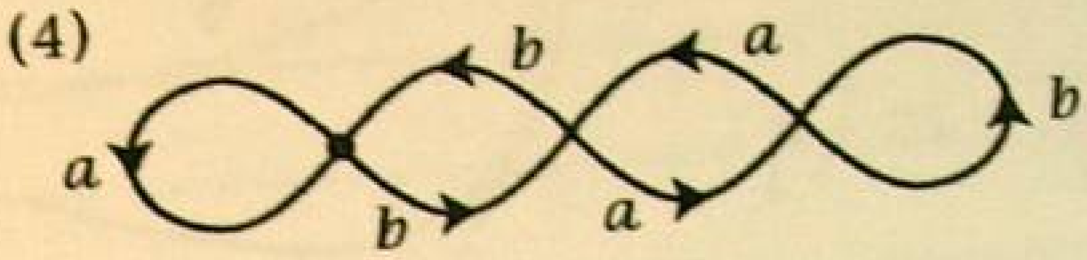
$\langle a^2, b^2, ab \rangle$

(3)



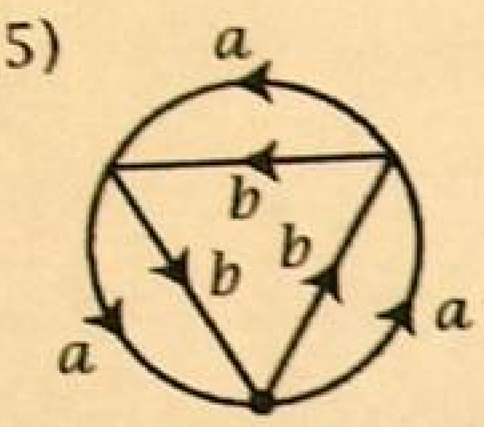
$\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$

(4)



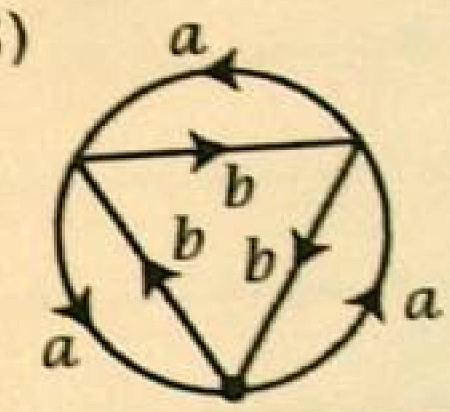
$\langle a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} \rangle$

(5)



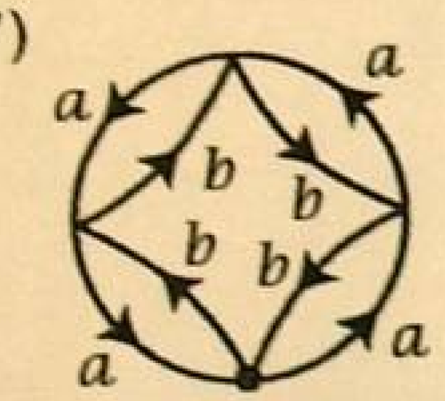
$\langle a^3, b^3, ab^{-1}, b^{-1}a \rangle$

(6)



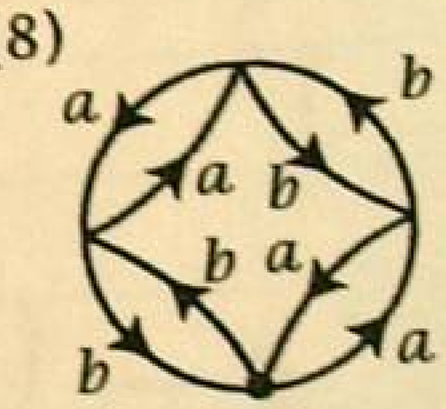
$\langle a^3, b^3, ab, ba \rangle$

(7)



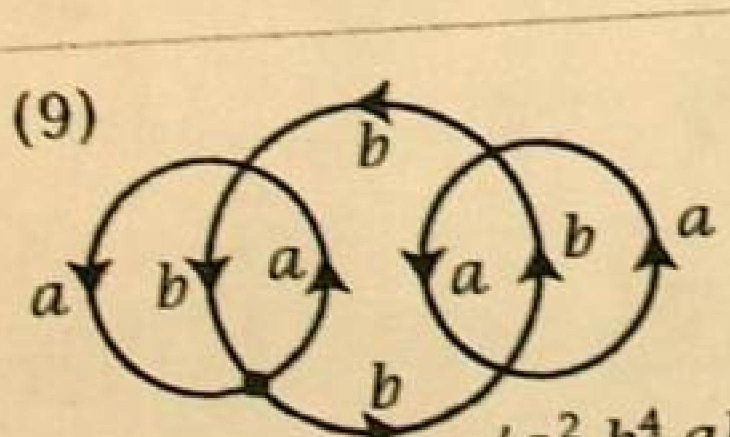
$\langle a^4, b^4, ab, ba, a^2b^2 \rangle$

(8)



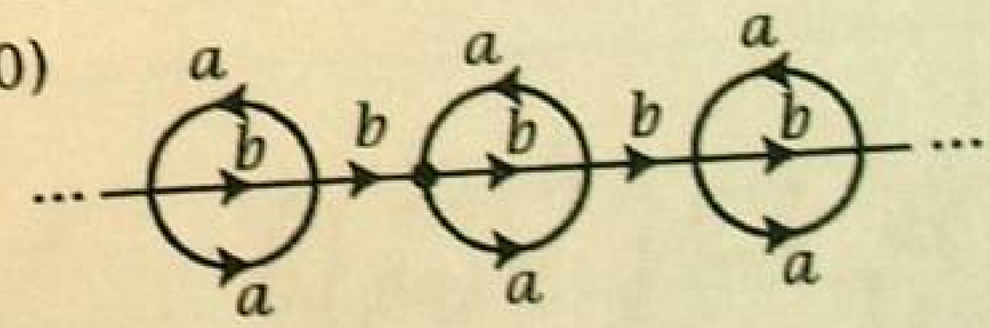
$\langle a^2, b^2, (ab)^2, (ba)^2, ab^2a \rangle$

(9)



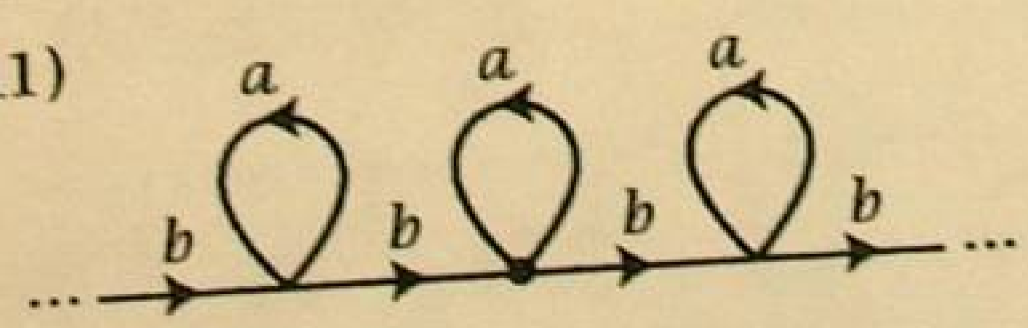
$\langle a^2, b^4, ab, ba^2b^{-1}, bab^{-2} \rangle$

(10)



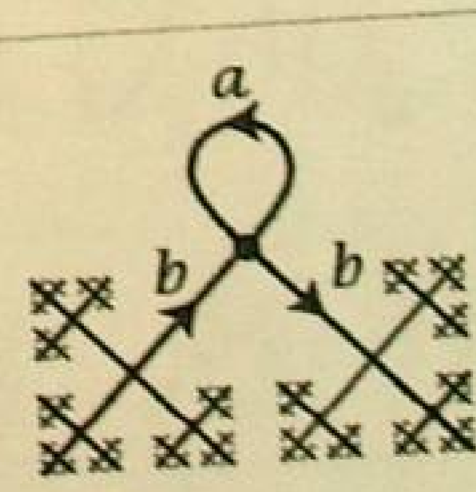
$\langle b^{2n}ab^{-2n-1}, b^{2n+1}ab^{-2n} \mid n \in \mathbb{Z} \rangle$

(11)



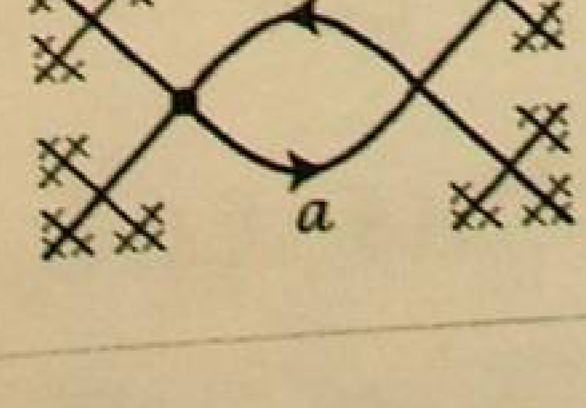
$\langle b^nab^{-n} \mid n \in \mathbb{Z} \rangle$

(12)



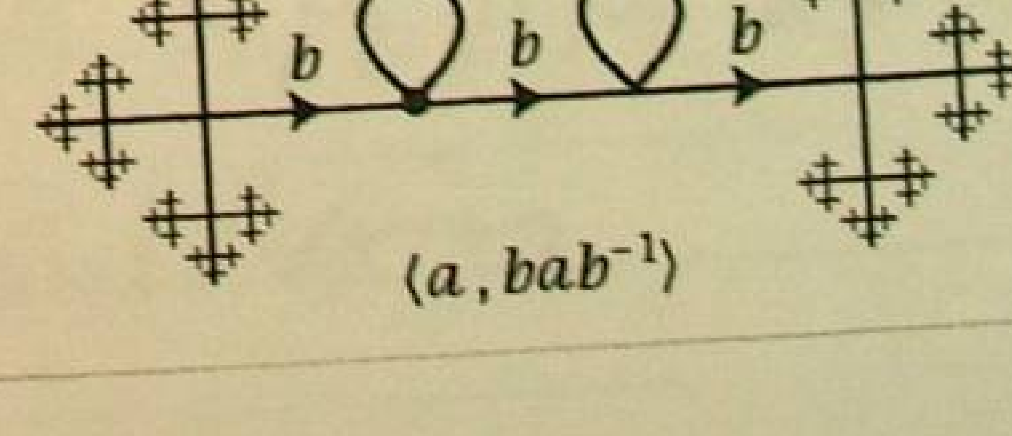
$\langle a \rangle$

(13)



$\langle ab \rangle$

(14)



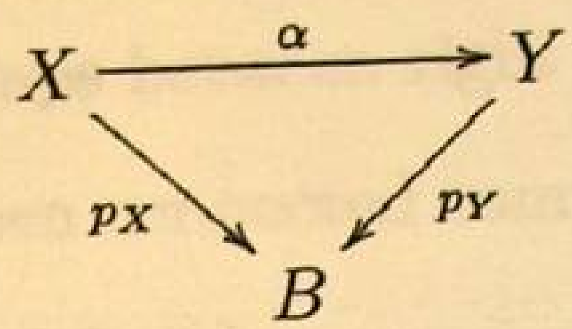
$\langle a, bab^{-1} \rangle$

Hatcher's book, page 58.

Covering Spaces in One Swoosh

web version: http://www.math.toronto.edu/~drorbn/classes/0405/Topology/CoveringSpaces/CoveringSpaces.html

Let B be a topological space and let $\mathcal{C}(B)$ be the category of covering spaces of B: The category whose objects are coverings $X \to B$ and whose morphisms are maps between such coverings that commute with the covering projections — a morphism between $p_X : X \to B$ and $p_Y : Y \to B$ is a map $\alpha : X \to Y$ so that the diagram



is commutative.

Every topologists' highest hope is to find that her/his favourite category of topological objects is equivalent to some category of easily understood algebraic objects. The following theorem realizes this dream in full in the case of the category $\mathcal{C}(B)$ of covering spaces of any reasonable base space B:

Theorem 1 (Classification of covering spaces) • *If B is connected and locally connected with base point b and fundamental group $G = \pi_1(B, b)$, then the map which assigns to every covering $p : X \to B$ its fiber $p^{-1}(b)$ over the basepoint b induces a functor $\mathcal{F}$ from the category $\mathcal{C}(B)$ of coverings of B to the category $\mathcal{S}(G)$ of G-sets — sets with a right G-action and set maps that respect the G action.*

- *If in addition B is semi-locally simply connected then the functor $\mathcal{F}$ is an equivalence of categories. (In fact, this is iff).*

If indeed the categories $\mathcal{C}(B)$ and $\mathcal{S}(G)$ are equivalent, one should be able to extract everything topological about a covering $p : X \to B$ from its associated G-set $\mathcal{F}(X) = p^{-1}(b)$. The following theorem shows this to be right in at least two ways:

Theorem 2 • *The set of connected components of X is in a bijective correspondence with the set of orbits of G in $\mathcal{F}(X)$.*

- *Let $x \in \mathcal{F}(X) = p^{-1}(b)$ be a basepoint for X that covers the basepoint b of B. Then the fundamental group $\pi_1(X, x)$ is isomorphic via the projection $p_\star$ into $G = \pi_1(B, b)$ to the the stabilizer group $\{h \in G : xh = x\}$ of x in $\mathcal{F}(X)$.*

(Both assertions of this theorem can be sharpened to deal with morphisms as well, but we will not bother to do so).

Ok. Every math technician can spend some time and effort and understand the statements and (only then) the proofs of these two theorems. Your true challenge is to digest the following statement:

All there is to know about covering spaces follows from these two theorems

In particular, the following facts are all simple algebraic corollaries of these theorems:

Corollary 3 *If X is connected then its covering number (= "number of decks") is equal to the index of $H = p_\star \pi_1(X)$ in $G = \pi_1(B)$, and the decks of X are in a non-canonical correspondence with the left cosets $H \backslash G$ of H in G.*

Corollary 4 *If B is semi-locally simply connected, there exists a unique (up to base-point-preserving isomorphism) "universal covering space U of B" (a connected and simply connected covering U).*

Corollary 5 *The group of automorphisms of the universal covering U is equal to $G = \pi_1(B)$.*

Corollary 6 $\pi_1(S^1) = \mathbb{Z}$.

Corollary 7 $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$.

Corollary 8 *If B is semi-locally simply connected, then for every $H < G = \pi_1(B)$ there is a unique (up to base-point-preserving isomorphism) connected covering space X with $p_\star \pi_1(X) = H$.*

Corollary 9 *If X_i for $i = 1, 2$ are connected coverings of B with groups $H_i = p_{i\star}\pi_1(X_i)$ and if $H_1 < H_2$ then X_1 is a covering of X_2 of covering number $(H_2 : H_1)$.*

Corollary 10 *If B is semi-locally simply connected there is a bijection between conjugacy classes of subgroups of $G = \pi_1(B)$ and unbased connected coverings of B.*

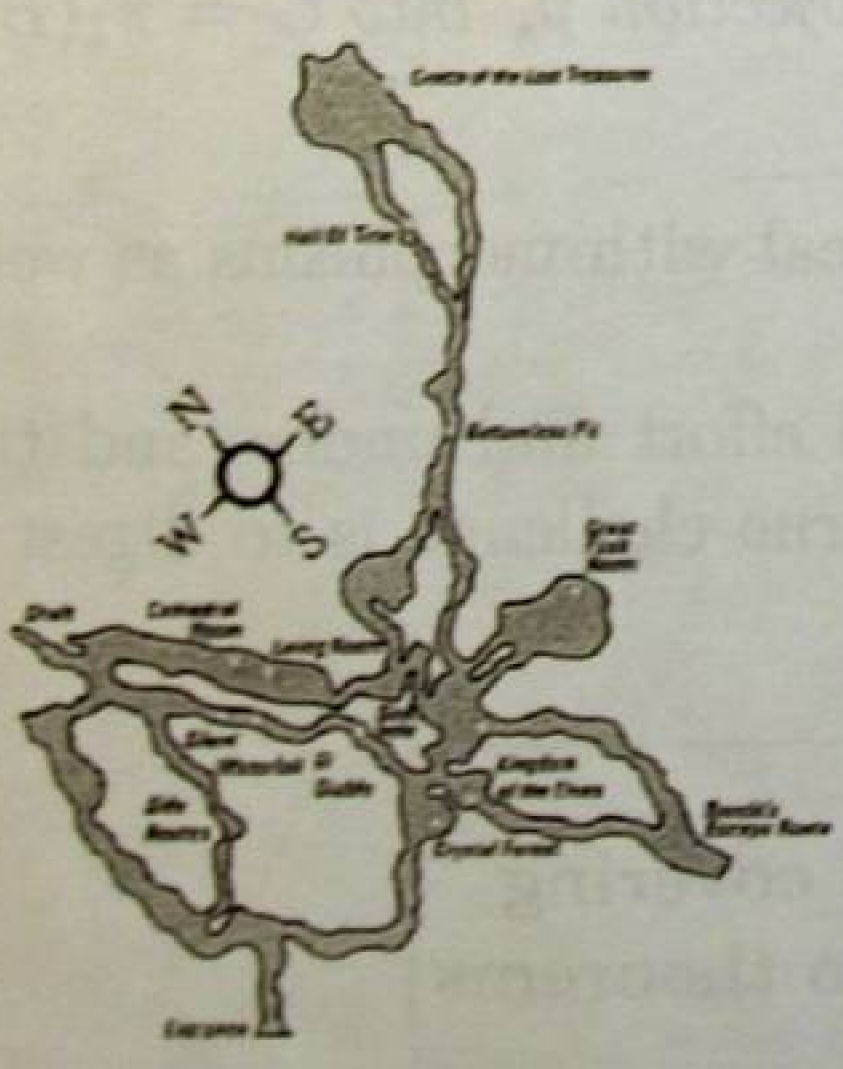
Corollary 11 *A connected covering X is normal (for any $x_1, x_2 \in p^{-1}(b)$ theres an automorphism τ of X with $\tau x_1 = x_2$) iff its group $p_\star \pi_1(X)$ is normal in $G = \pi_1(B)$.*

Corollary 12 *If X is a connected covering of B and $H = p_\star \pi_1(X)$, then $\mathrm{Aut}(X) = N_G(H)/H$ where $N_G(H)$ is the normalizer of H in G.*

Proposition 13 *If I forgot anything, it follows too.*

Steps in the proofs of Theorem 1 and 2.

1. Use path liftings to construct a right action of G on $p^{-1}(b)$.
2. Show that this is indeed a group action and that morphisms of coverings induce morphisms of right G-sets.



A map of Colossal Cave, New Mexico, from http://www.colossalcave.com/cavetour.html.

3. Start the construction of an "inverse" functor $\mathcal{G}$ of $\mathcal{F}$: Use spelunking (cave exploration) to construct a universal covering U of B, if B is semi-locally simply connected.

4. Show that $\mathcal{F}(U) = G$.

5. Use the construction of U or the general lifting property for covering spaces to show that there is a left action of G on U.

6. For a general right G-set S set $\mathcal{G}(S) = S \times_G U = \{(s,u) \in S \times U\}/(sg,u) \sim (g,su)$ and show that $\mathcal{G}(S)$ is a covering of B and $\mathcal{F}(\mathcal{G}(S)) = S$.

7. Show that $\mathcal{G}$ is compatible with maps between right G-sets.

8. Understand the relationship between connected components and orbits.

9. Prove Theorem 2.

10. Use the existence and uniqueness of lifts to show that $\mathcal{G} \circ \mathcal{F}$ is equivalent to the identity functor (working connected component by connected component).

A Deep Thought Question. We'll get there when it's time, but meanwhile, think on your own: What does it at all mean "$\mathcal{G} \circ \mathcal{F}$ is equivalent to the identity functor" (and first, why can't it simply **be** the identity functor)? And even harder, what does it at all mean for two categories to be "equivalent"? If you answer this question correctly, you'll probably re-invent the notions of "natural transformation between two functors" and "natural equivalence", that gave the historical impetus for the development of category theory.

Category theory
From Wikipedia, the free encyclopedia (http://en.wikipedia.org/wiki/Category_theory).

Background

A category attempts to capture the essence of a class of related mathematical objects, for instance the class of groups. Instead of focusing on the individual objects (groups) as has been done traditionally, the morphisms — i.e. the structure-preserving maps between these objects — are emphasized. In the example of groups, these are the group homomorphisms. Then it becomes possible to relate different categories by functors, generalizations of functions which associate to every object of one category an object of another category and to every morphism in the first category a morphism in the second. Very commonly, certain "natural constructions", such as the fundamental group of a topological space, can be expressed as functors. Furthermore, different such constructions are often "naturally related" which leads to the concept of natural transformation, a way to "map" one functor to another. Throughout mathematics, one encounters "natural isomorphisms", things that are (essentially) the same in a "canonical way". This is made precise by special natural transformations, the natural isomorphisms.

Historical notes

Categories, functors and natural transformations were introduced by Samuel Eilenberg and Saunders Mac Lane in 1945. Initially, the notions were applied in topology, especially algebraic topology, as an important part of the transition from homology (an intuitive and geometric concept) to homology theory, an axiomatic approach. It has been claimed, for example by or on behalf of Ulam, that comparable ideas were current in the later 1930s in the Polish school.

Eilenberg/Mac Lane have said that their goal was to understand natural transformations; in order to do that, functors had to be defined; and to define functors one needed categories.

The subsequent development of the theory was powered first by the computational needs of homological algebra; and then by the axiomatic needs of algebraic geometry, the field most resistant to the Russell-Whitehead view of united foundations. General category theory — an updated universal algebra with many new features allowing for semantic flexibility and higher-order logic — came later; it is now applied throughout mathematics.

Special categories called topoi can even serve as an alternative to axiomatic set theory as the foundation of mathematics. These broadly-based foundational applications of category theory are contentious; but they have been worked out in quite some detail, as a commentary on or basis for constructive mathematics. One can say, in particular, that axiomatic set theory still hasn't been replaced by the category-theoretic commentary on it, in the everyday usage of mathematicians. The idea of bringing category theory into earlier, undergraduate teaching (signified by the difference between the Birkhoff-Mac Lane and later Mac Lane-Birkhoff abstract algebra texts) has hit noticeable opposition.

Categorical logic is now a well-defined field based on type theory for intuitionistic logics, with application to the theory of functional programming and domain theory, all in a setting of a cartesian closed category as non-syntactic description of a lambda calculus. At the very least, the use of category theory language allows one to clarify what exactly these related areas have in common (in an abstract sense).

Errata to Munkres' Book

```
From [email suppressed] Tue Dec 14 18:46:06 2004
Date: Thu, 9 Dec 2004 22:10:01 -0500
From: Barbara and Jim Munkres [email suppressed]
To: drorbn@math.toronto.edu
Subject: I hope this is useful

ERRATA FOR TOPOLOGY, SECOND EDITION  (second and subsequent printings)

xii, 13      of connectedness and compactness in Chapter 3.

107; 2      f maps [0,1) into S super 1

118; Exercise 9, line 2,          J is not empty.

143; 1     composite  g  is ...

151; 2*     (a sub 1, ..., a sub N, 0, 0, ...)

187; 4*     Let  A  be a subset of  X.

203; 12     b < a.  Neither  U  nor  V  contains  a sub 0.

203; 15     ... U  and  V  not containing  a sub 0, but containing

205; 9*     if and only if  X  is  T sub 1  and for every...

224;  13     open in  X sub i  for each  i.

235; 13*     Show that if  X  is Hausdorff

237; 8      Assume script A  is a covering of  X  by basis elements
such that

251;  7     less than or equal to  1/n

261; 7      replace "paracompact" by "metrizable".

262; 8      (x, epsilon sub i)

263; 1*     Throughout, we assume Section 28.

266; 8*     rho super bar  is a metric;

356; 7     Find a ball centered at the origin...

417; 11     element of P(W),

421; 8     length (at least 3), then

425; 10*     (G sub 1) * (G sub 2)

445; 10     Exercise 2 should be starred.

466; 4     =  (w sub 0)[y sub 1] a [y sub 2] b ...

481; 1     with k(h(e sub 0)) = e sub 0.

488; 4     F = p inverse (b sub 0).

488; 11     of the subset

503; 14*     either empty or a one- or two- point set!
```

Math 1300 Topology, Tue Jan 4 2005

1. Go over "covering of ∞" handout
   - * explain the covers
   - * "Normal"
   - * covers of covers [maps between covers]

theorems

2. Thm connected covers of $B \Longleftrightarrow$ subgroups of $\pi_1(B)$

   via $\begin{matrix} X \\ \downarrow p \\ B \end{matrix} \longrightarrow p_*\pi_1(X)$

   and then prove many theorems about automorphisms, covers of covers.

3. Our goal: one big swoosh.

   1. Define $\mathcal{C}(B)$
   2. Define $\mathcal{S}(G)$

   State theorem 1

   (state how $\pi_1(B)$ acts on $p^{-1}(b)$)

   State theorem 2

4. Corollaries 3, 4, 5, 6, 7

(familiarize: G = Aut of

G = Aut(cube)

S_1 = verts S_3 = fac

S_2 = edges S_4 = pairs of two

• orbits, transitivity

stabilizer,

claim If $X \circlearrowleft G$ is t

& $x \in X$, then $X \cong H\backslash$

where $H = Stab(x)$

ALGEBRAIC TOPOLOGY, PROBLEM SET 3

1. Fundamental group of knot complement

In the following exercises a knot is a piecewise linear or smooth embedding of S^1 into $\mathbb{R}^3$ or S^3.

Exercise 1.1. Let $K \subset S^3$ be a knot. Consider the image of K in $\mathbb{R}^3$ under the stereographic projection $\sigma : S^3 \setminus \{\mathrm{pt}\} \to \mathbb{R}^3$. Prove that $\pi_1(S^3 \setminus K, x_0) \cong \pi_1(\mathbb{R}^3 \setminus \sigma(K), \sigma(x_0))$.

Exercise 1.2. Compute the presentation of a fundamental group of the complement of a knot $K \subset \mathbb{R}^3$. (You may work along the lines of Exercise 22, p. 55 in the Hatcher's book).

Exercise 1.3. Compute, using the above exercise, a presentation of the trefoil knot (this is the $(3,2)$-toric knot) and find an isomorphism between the two different presentation of the fundamental group of the complement of the trefoil: as an abstract knot and as a toric knot ($< a, b \mid a^2b^3 >$). (*Hint:* It may be not easy as an algebraic problem, so use the picture!)

2. Covering maps. First examples.

Solve Exercises 1, 2, 3, 4 from page 79. In the Exercise 4 skip the second part (or solve it as an optional exercise, but first read Example 1.45, p. 77).

Date: April 11, 2002.

ALGEBRAIC TOPOLOGY, PROBLEM SET 4

1. Covering maps. Applications of the lifting theorems.

Solve Exercises 8, 9 from page 79. Use various lifting properties you've recently learnt. The hint for the Exercise 8 in the Hatcher's book seems misleading. Note, that a map is called nullhomotopic if it is homotopic to a map into a point. Or, equivalently, a map is nullhomotopic if it factors through a contractible space (Why ?).

2. More properties of covering maps.

Exercise 2.1. Let G be a connected and locally linearly connected topological group (a topological space G with a distinguished point $e \in G$ and two continuous maps: multiplication $\mu : G \times G \to G$ and inverse $\nu : G \to G$ satisfying obvious axioms). And let $(\tilde{G}, p)$ be a covering space. Show that $\tilde{G}$ may be given a structure of topological group.

Exercise 2.2. Let G be a group which acts freely and discretely (this means that any point has a neighbourhood U such that all open sets $gU, g \in G$ are pairwise disjoint) on a space X. Then the natural projection $X \to X/G$ is a covering map.

Date: April 18, 2002.

ALGEBRAIC TOPOLOGY, PROBLEM SET 5

1. Covering maps: Action of the fundamental group of the base on the fiber

Exercise 1.1. Let $(\tilde{X}, p)$ be a covering space of X. Then the group of automorphisms of the covering space $A(\tilde{X}, p)$ is isomorphic to the group of automorphisms of a set $p^{-1}(x), x \in X$ considered as a $\pi_1(X, x)$-space.

Exercise 1.2 (Continued). For any two points $x \in X$ and $\tilde{x} \in p^{-1}(x)$

$$A(\tilde{X}, p) \cong N(p_* \pi_1(\tilde{X}, \tilde{x}))/p_* \pi_1(\tilde{X}, \tilde{x}),$$

where $N(\,\cdot\,)$ is the normalizer in $\pi_1(X, x)$.

What happens if $\pi_1(\tilde{X}, \tilde{x})$ is normal? Discuss the notion of a normal covering space. What can you say about the case of the universal covering space?

2. Applications

Solve Exercises 15, 17 from page 80.

Exercise 2.1. Find the fundamental group of a real projective space $\mathbb{R}P^n, n \geq 2$.

Date: April 25, 2002.

ALGEBRAIC TOPOLOGY, PROBLEM SET 6

1. Calculation of the fundamental group of $SO(3)$

$SO(3)$ is the group of automorphisms of the 3-dimensional Euclidian space which preserve an orientation. The goal of the current set of exercises is to understand $\pi_1(SO(3), I)$, where I is the identity matrix.

Exercise 1.1. Let E be the set of all traceless, anti-hermitian 2×2-matrices, i.e.

$$E = \{A \in M_2(\mathbb{C}) \mid \mathrm{Tr}(A) = 0; \bar{A}^t = -A\}.$$

Introduce on E a structure of the 3-dimensional Euclidian space. (*Hint:* No tricks ! Use the most natural structure you can think of.)

Exercise 1.2. $SU(2)$ is the set of all complex 2×2-matrices with determinant 1 which preserve 'one and a half'-linear form on $\mathbb{C}^2$. Equivalently,

$$SU(2) = \{U \in M_2(\mathbb{C}) \mid \bar{U}^t U = I, \det(U) = 1\}$$

Find a homeomorphism $SU(2) \cong S^3$.

Exercise 1.3. Find the action of $SU(2)$ on E which preserves the Euclidian structure and orientation. (*Hint:* Adjunction will work.)

Exercise 1.4. Explain why in the previous exercise we've constructed a homomorphism $p\colon SU(2) \to SO(3)$.

Exercise 1.5. Consider $\ker(p)$ from the previous exercise and conclude that p is a (universal) covering map.

Exercise 1.6. Find $\pi_1(SO(3)$.

Date: May 2, 2002.

Bel + ?

Math 1300 Topology, Thu Jan 6 2005

Recall $\mathcal{C}(B)$ $G = \pi_1(B, b)$

Recall $\mathcal{S}(G)$ (right G-sets)

<u>Thm 1</u> for reasonable B: $\mathcal{C}(B)$ is equivalent to $\mathcal{S}(G)$

via $\mathcal{F}\left(\begin{smallmatrix} X \\ \downarrow p \\ B \end{smallmatrix}\right) = p^{-1}(B)$ ⟸ explain how this is a G-set.

Every G-set is a union of transitive G-sets.

If S is a transitive G-set & $x \in S$,

then $S \cong H \backslash G$, $H = \mathrm{Stab}_S\, x = \{h : xh = x\}$

$H \backslash G = \{Hg : g \in G\}$

<u>Pf</u> $H \backslash G \longrightarrow S$ via $Hg \longmapsto xHg = xg$

<u>Thm 2</u> 1. Connected comps of X ⟺ orbits of $\mathcal{F}(X)$

2. $x \in \mathcal{F}(X) = p^{-1}(b) \Longrightarrow \pi_1(X, x) = \mathrm{Stab}(x)$

Belt ?

Tues

Math 300 Topology, ~~Wed~~ Tues Jan 11 2005

onboard: $\mathcal{E}(B)$ $\qquad\qquad$ $\mathcal{S}(G)$

$$X \longrightarrow Y, \quad p: X \to B, \quad q: Y \to B$$

$$\begin{array}{ccc} S_1 & \xrightarrow{\alpha} & S_2 \\ \downarrow \cdot g & & \downarrow \cdot g \\ S_1 & \longrightarrow & S_2 \end{array}$$

Thm The functor $\mathcal{F}: \mathcal{E}(B) \longrightarrow \mathcal{S}(G = \pi_1(B))$

Given by $X \longmapsto p^{-1}(b)$ is an equiv. of categories; furthermore,

a. connected components correspond to orbits

b. $p_*: \pi_1(X, x) \simeq \mathrm{stab}_{\mathcal{F}(X)}(x)$

Cor 4 $\exists$ "universal covering space" U

Cor 5 $\mathrm{Aut}(U) = G$

Cor 6 $\pi_1(S^1) = \mathbb{Z}$

Cor 7 $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$ + Belt & glass tricks.

$SO(3) = \left\{ A \in M_{3\times 3}(\mathbb{R}) : \begin{array}{c} AA^T = I \\ \det A = 1 \end{array} \right\}$ $\qquad$ $SU(2) = \left\{ U \in M_{2\times 2}(\mathbb{C}) : \begin{array}{c} U\bar{U}^T = I \\ \det U = 1 \end{array} \right\}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $= \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} : |z|^2 + |w|^2 = 1 \right\} = S^3$

$su(2) = \left\{ H \in M_2(\mathbb{C}) : \begin{array}{c} H + \bar{H}^T = 0 \\ \mathrm{tr}\, H = 0 \end{array} \right\}$

$= \left\{ \begin{pmatrix} it & a \\ -\bar{a} & -it \end{pmatrix} : \begin{array}{c} t \in \mathbb{R} \\ a \in \mathbb{C} \end{array} \right\} = \mathbb{R}^3$

claim $U \overset{p}{\longmapsto} (H \to U^{-1} H U)$ defines $p: SU(2) \to SO(3)$

---

on to the proof of the main thm !

Math 1300 Topology, Thurs. Jan 13 2005

Continue the proof of the main theorem:

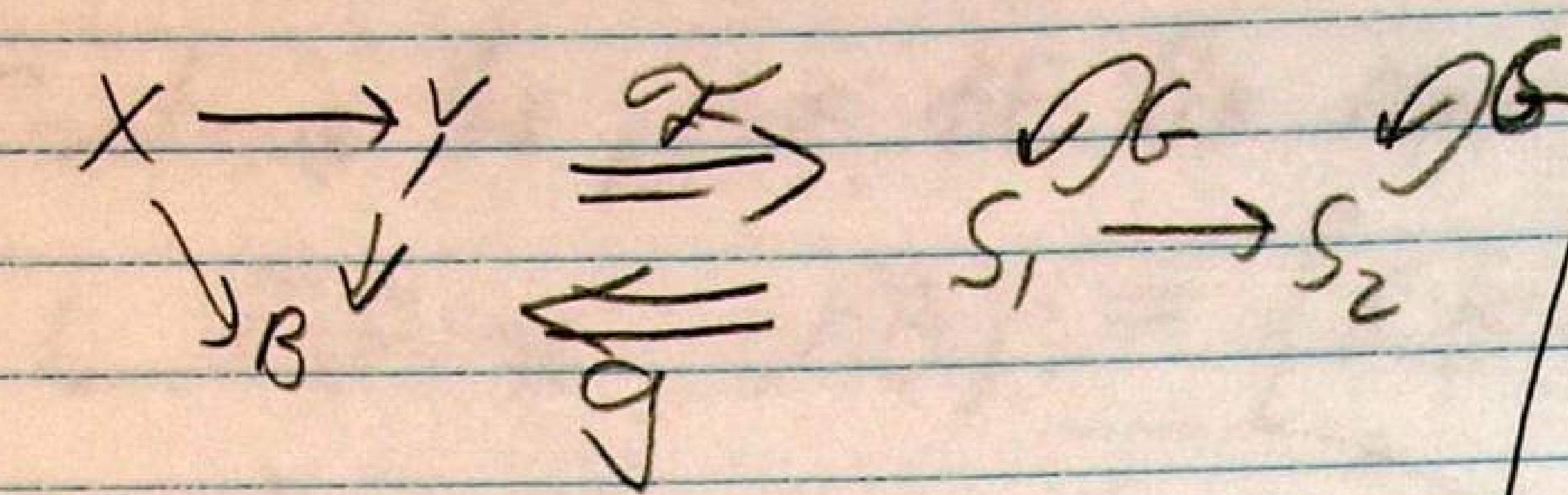
1. Define $U(B) := \{\gamma : [0,1] \to B : \gamma(0) = b\} / $ end-point preserving homotopies
2. Topologize $U(B)$ — $p : U(B) \to B$ by $\gamma \mapsto \gamma(1)$
3. p is a local homeomorphism (use c., l.c., s.l.s.c)
4. $p^{-1}(b) = G$ $\quad p^{-1}(b') \cong G$
5. p is a covering map
6. $U(B)$ is connected and simply connected.

   Lemma if $p : X \to B$ is a covering map, the $p_* : \pi_1(X) \to \pi_1(B)$ is injective

7. There is a left action of G on U.

Math 1300 Topology, Tues. Jan 18 2005

on board:



Thm These categories are equivalent.

Pf Enough! I want home! I want homology! Ok, here are hints for what remains.

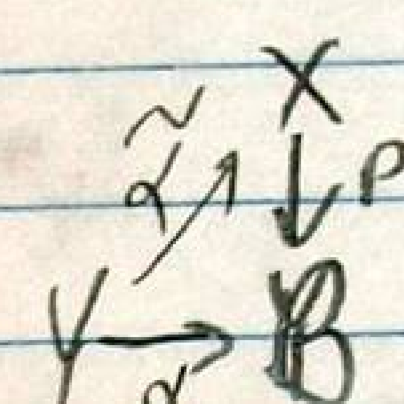
Hints

1. U by splunting

   $\pi_1(U)$ simply connected:

2. Construct $\mathcal{G}$ from U

3. natural transformations & natural equivalences.

4. 

   when does $\tilde{\alpha}$ exist? Is it unique?

The intuitive idea of homology, $H_*(S^2)$, T^2, S^n, $\mathbb{RP}^2$

~~Realization as on May 7, 2002~~

Math 1300 Topology, Thu. Jan 20 2005

Follow May 7-9, 2002.

Goals: 1. YPIS
2. Homology lite
3. Bigwords: "Homotopy invariance of homology".

Math 1300 Topology, Tue Jan 25 2005

on board:

$\Delta^n = \{ \bar{t} \in \mathbb{R}^{n+1} : \sum_0^n t_i = 1 \}$ $\quad [v_0, \ldots v_n] : \Delta^n \to V$, $\bar{t} \mapsto \sum t_i v_i$

$C_n(X) = \langle \sigma : \Delta^n \to X \rangle$ $\quad \partial_n : C_n \to C_{n-1}$ $\quad \sigma \mapsto \sum (-1)^i \sigma \circ [\ldots \hat{e}_i \ldots]$

<u>Lemma</u> $\partial^2 = 0$ "$\to C_{n+1} \to C_n \to C_{n-1} \to \ldots$ is a complex"

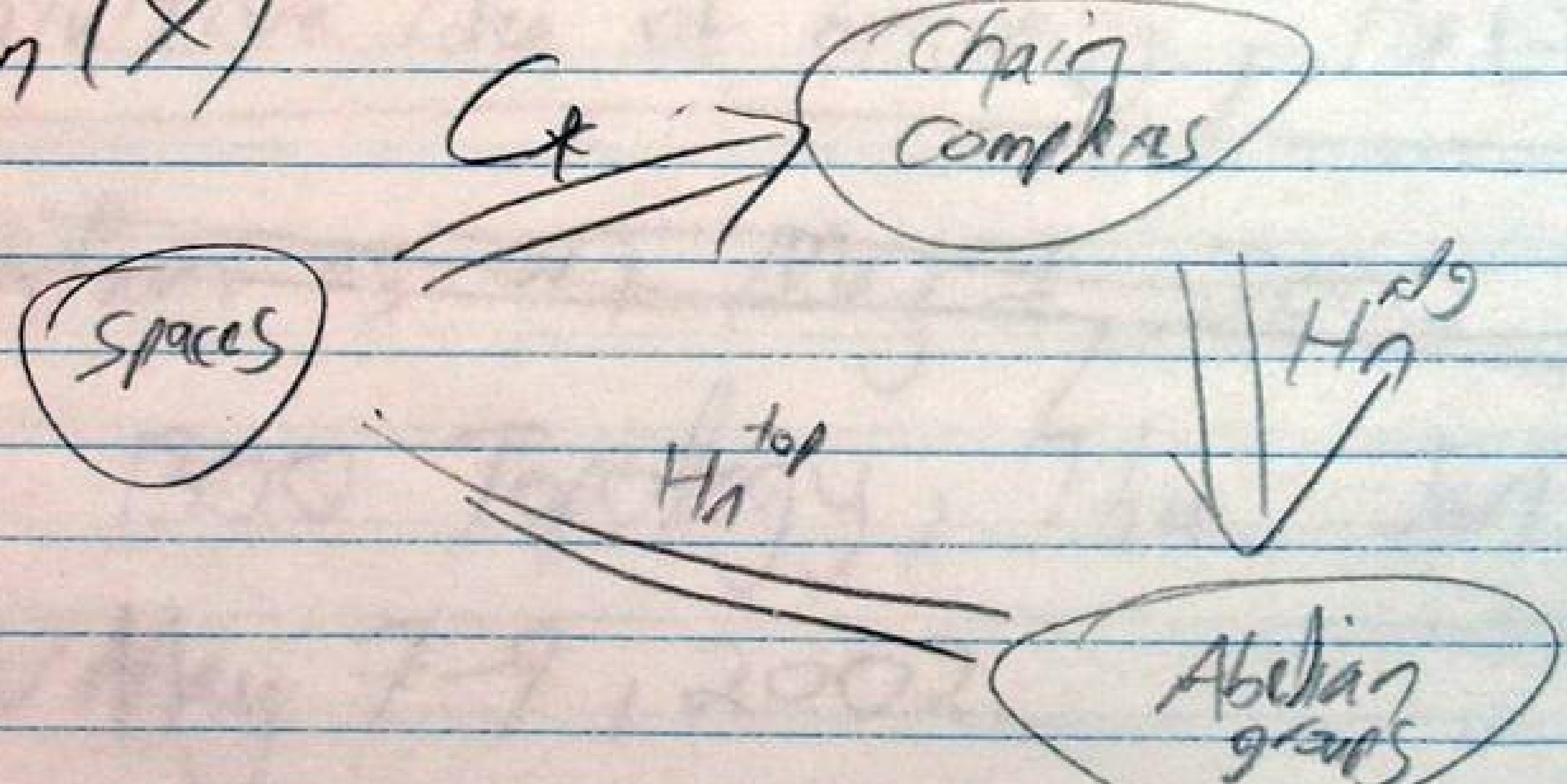
$Z_n = \ker \partial_n$ $\quad B_n = \operatorname{im} \partial_{n+1}$ $\quad H_n \stackrel{def}{=} Z_n / B_n$

YPIS pf of $\partial^2 = 0$

HomLite: $H_n(pt)$, $X = \bigcup X_i \Rightarrow H_n(X) = \bigoplus H_n(X_i)$

$H_0(X) = \mathbb{Z}^{\#(\text{connected comps})}$

$\tilde{H}_n(X)$



lift to a diagram of functors.

Homotopy invariance of homology:

<u>Thm</u> $f \sim g : X \to Y \Rightarrow (H_n(f) = H_n(g)) : H_n(X) \to H_n(Y)$

<u>Cor</u> $X \sim Y \Rightarrow H_n(X) = H_n(Y)$; $H_*(\mathbb{R}^k) = \ldots$

Intuitive idea, realization:

$$P\sigma = \sum_{i=0}^{n} (-1)^i H \circ (\sigma \times Id) \circ [f_0, \ldots f_i g_i g_{i+1} \ldots g_n]$$

<u>Claim</u> $\partial P \sigma - P \partial \sigma = f_* \sigma - g_* \sigma$

Math 1300 Topology, Thu Jan 27 2005

(on board)

$f \sim g : X \to Y$ i.e. $H : X \times I \to Y$, $H|_{t=0} = f$ $H|_{t=1} = g$

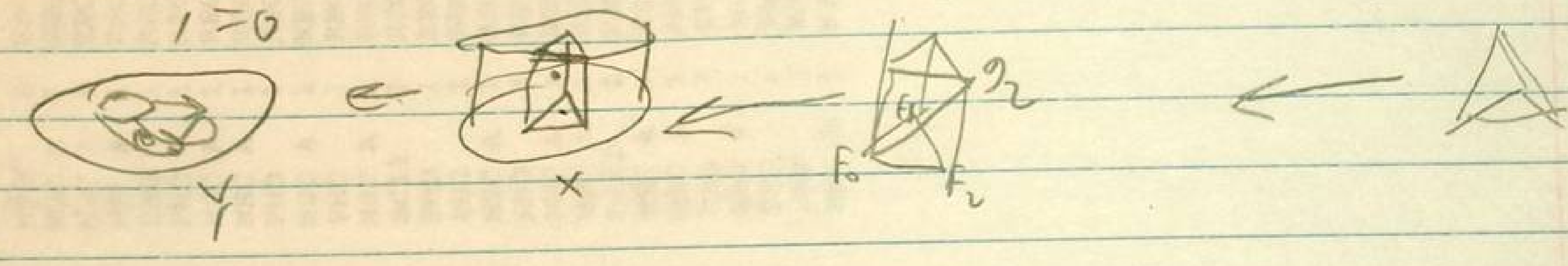
$$\begin{array}{ccccccc} \cdots C^{n+1}(X) & \xrightarrow{\partial} & C^n(X) & \xrightarrow{\partial} & C^{n-1}(X) & \cdots \\ f_* \downarrow \downarrow g_* & \swarrow^{P} & f_* \downarrow \downarrow g_* & \swarrow^{P} & f_* \downarrow \downarrow g_* & \\ \cdots C^{n+1}(Y) & \xrightarrow{\partial} & C^n(Y) & \xrightarrow{\partial} & C^{n-1}(Y) & \cdots \end{array}$$

$\partial\sigma = \sum (-1)^i \sigma \circ [e_0 \ldots \hat{e}_i \ldots e_n]$

dream: $\partial P \sigma - P \partial \sigma = f_* - g_*$

$P\sigma = \sum_{i=0}^{n} (-1)^i H \circ (\sigma \times Id) \circ [f_0 \ldots f_i g_i g_{i+1} \ldots g_n]$

start here



<u>claim</u> Indeed $\partial P - P\partial = f_* - g_*$

Continue as on May 14, 2002

* If $A \subset X$ is non-empty & closed, and a deformation retract of a nbhd neiborhood of it in X, then there is a "long exact sequence"

$$\longrightarrow \tilde{H}_n(A) \longrightarrow \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X/A) \longrightarrow \tilde{H}_{n-1}(A) \to \ldots$$

<u>def</u>
1. def. retract.
2. Top on X/A: a the biggest for which $X \xrightarrow{\pi} X/A$ is cont.

   Aside X/A could be ugly

   $X \xrightarrow{\tilde{f}} Y$, $\pi \downarrow$, $X/A \xrightarrow{f} Y$: π cont, $\tilde{f}$ cont $\Rightarrow f$ cont
3. exact sequences:

   $0 \to A \to 0$, $0 \to A \to B$, $A \to B \to 0$,
   $0 \to A \to B \to 0$, $0 \to A \to B \to C \to 0$
4. $A \xrightarrow{i} X \xrightarrow{j} X/A$
5. Compute $\tilde{H}_*(S^n)$ using $(X, A) = (D^n, S^{n-1})$

Math 1300 Topology, Tue Feb 1, 2005

on board.

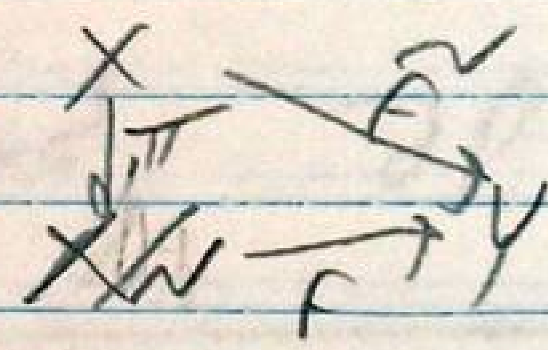
"A short exact sequence of spaces yields a long exact sequence of homology groups".

If $A \subset X$ is non-empty & closed & a deformation retract of a nbd thereof, then there is a long exact sequence

$$\tilde{H}_{n+1}(X/A) \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A)$$

Reminder: $0 \to A \xrightarrow{\varphi} B \to 0$ exact: φ an iso

$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$ exact: $C = B/A$

1. def retract
2. X/A: 

, can be problematic.
3. $A \overset{i}{\hookrightarrow} X \xrightarrow{p} X/A$ is "exact".
4. Compute $\tilde{H}_*(S^n)$ using $(X,A) = (D^n, S^{n-1})$
5. The Brouwer fixed point theorem
6. Algebraic analog: a short exact sequence of complexes yields a long exact sequence of homology groups

   pf by diagram chase
7. $H_n(X,A)$, the long exact sequence.
8. $\tilde{H}_n(X,A)$; $\tilde{H}_n(X) \cong H_n(X,pt)$
9. The category of pairs, homotopy of pairs.
10. The long exact sequence of a triple $B \subset A \subset X$

טופולוגיה אלגברית - 23 למאי 2002

<u>משפט</u> לסדרה מדויקת קצרה של מרחבים

$$\phi \to A \xrightarrow{i} X \xrightarrow{j} X/A \to \phi$$

(כאשר A סגורה ונסג דפורמציה של סביבה שלה) מתאימה סדרה מדויקת ארוכה של הומולוגיות

$$\dots \to \tilde{H}_{n+1}(A) \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \xrightarrow{\partial} \tilde{H}_{n-1}(A) \to \dots$$

$H_k(S^n) =$ מסקנה

~~<u>תזכורת מאלגברה</u>: לסדרה מדויקת קצרה של קומפלקסים $0 \to A_* \to B_* \to C_* \to 0$ מתאימה ...~~

<u>הגדרת</u> $H_n(X,A)$: פונקטור מקטגורית הזוגות אל חבורות אבליות, אינווריאנטי תחת הומוטופיה ומתאים שקבלו הומומורפיזם, הסדרה המדויקת הארוכה.

<u>דוגמאות</u>: $H_k(D^n, S^{n-1})$

הסדרה המדויקת הארוכה עבור הומולוגיה מצומצמת

השוויון $H_n(X, pt) = \tilde{H}_n(X)$

רעיון ההוכחה

$C_*(X/A, A/A) \leftarrow C(X,A)$

ההשוואה דרך:

$$\begin{array}{ccccc} H_n(X,A) & \longrightarrow & H_n(X,V) & \longleftarrow & H_n(X-A, V-A) \\ \downarrow & & \downarrow & & \downarrow \\ H_n(X/A, A/A) & \longrightarrow & H_n(X/A, V/A) & \longleftarrow & H_n(X/A - A/A, V/A - A/A) \end{array}$$

<u>משפט</u> אם $Z \subset A \subset X$ וסגור Z מוכל בפנים A, אז:

$H_n(X-Z, A-Z) \to H_n(X,A)$ איזומורפיזם.

נוסחאות לחלוקה מרכזית:

$$b[v_0 \ldots v_n] = [b, v_0 \ldots v_n]$$

$$b_\lambda = \sum v_i / n+1$$

$$\lambda = [v_0 \ldots v_n]$$

$$\begin{cases} S\lambda = b_\lambda(S\partial\lambda) \\ S[\phi] = [\phi] \end{cases}$$

אם $\sigma \in S\lambda$ אז $\operatorname{diam}\sigma \leq \frac{n}{n+1}\operatorname{diam}\lambda$

טענה

$$\partial S = S\partial$$

$$T\lambda = \begin{cases} 0 & \lambda \in LC_{-1}(Y) \\ b_\lambda(\lambda - T\partial\lambda) \end{cases}$$

טענה

$$\partial T\lambda + T\partial\lambda = Id - S$$

Math 1300 Topology, Thu Feb 3, 2005

on board

Given $A \hookrightarrow X \twoheadrightarrow X/A$ (A closed & a def. retract of a nbd)

want $\tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \tilde{H}_{n-1}(A)$

Have $H_n(A) \to H_n(X) \to H_n(X,A) \xrightarrow{\partial} H_{n-1}(A)$ (from $0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$)

need $H_n(X,A) = \tilde{H}_n(X/A)$ (and ~ nonsense)

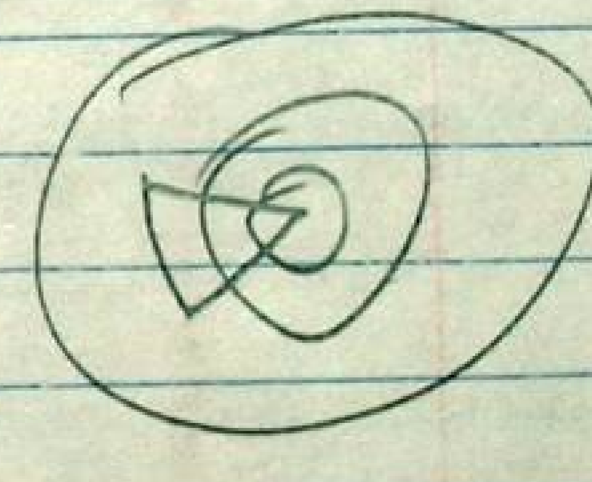
1. $\tilde{H}_n(X,A) = H_n(X,A)$; $\tilde{H}_n(X) = H_n(X, pt)$

2. The category of pairs, homotopy of pairs.

3. The long exact sequence of a triple $B \subset A \subset X$.

4. Excision: $Z \subset A \subset X$, $\bar{Z} \subset A^\circ \Rightarrow$

$$H_n(X-Z, A-Z) \xrightarrow{\sim} H_n(X,A)$$


reasonable but non-trivial.

5.

$$\begin{array}{ccccc} H_n(X,A) & \xrightarrow{\sim} & H_n(X,V) & \xleftarrow{\sim} & H_n(X-A, V-A) \\ \downarrow & & \downarrow & & \downarrow \wr \\ \tilde{H}_n(X/A) = H_n(X/A, A/A) & \xrightarrow{\sim} & H_n(X/A, V/A) & \xleftarrow{\sim} & H_n(X/A - A/A, V/A - A/A) \end{array}$$

6. $H_n^u(X)$, $H_n^u(X,A)$ Prop $H_n^u(X) \xrightarrow{\sim} H_n(X)$

$H_n^u(X,A) \xrightarrow{\sim} H_n(X,A)$

7. Proof of excision

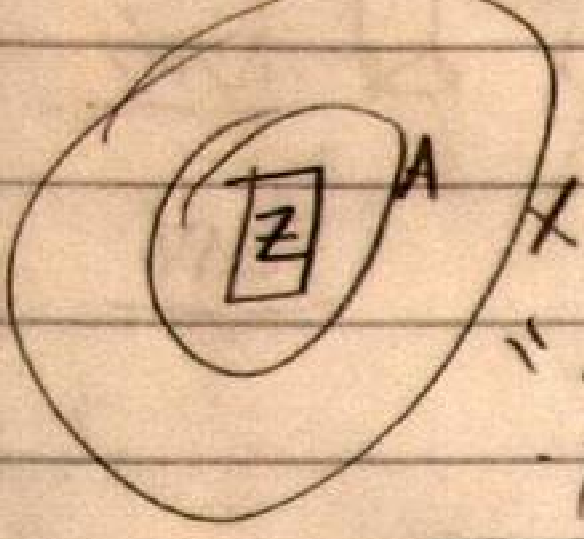
$\mathcal{U} = \{A^\circ, X \setminus \bar{Z}\}$ $C_n^u(X,A) \longleftarrow C_n(X-Z, A-Z)$

רשימות חסרות: 2/5, ~~[illegible]~~, 14/5, ~~[illegible]~~

טופולוגיה אלגברית - 28 למאי 2002

<u>משפט ההרחקה</u>: אם $Z \subset A \subset X$ וסגור Z מוכל בפנים A, אזי

$H_n(X-Z, A-Z) \to H_n(X,A)$ איזומורפיזם



״אם שואה דבר מה שאינו גורם בתחום אז שוכחים אותו״

נוסחאות לחלוקה: שרטוט כקלט הגדרה.

<u>הערות</u>: 1. מחשבים את מחלקות ההומולוגיה של $H^n(S^n)$, $H^n(D^n, S^{n-1})$

2. אם $U \subset \mathbb{R}^n$ פתוחה הומאומורפית ל- $V \subset \mathbb{R}^m$ פתוחה אז $m=n$

3. טבעיות של הסדרה המדויקת הארוכה.

<u>הגדרה</u> מרחב סימפליציאלי: אוסף של סימפלקסים סגורים יחד עם זיהוי סדור של כל אחת מ $(n+1)$ פאות השפה של כל אחד מהם שמזהה n פאותיו עם אחד מהסימפלקסים ה $(n-1)$ מימדיים שבאוסף.

<u>דוגמאות</u>: T, $\mathbb{RP}^2$, K

<u>הערה</u>: זהו לא מבנה קומבינטורי ולא מרחב טופולוגי. למרות מרחב X, $|X|$

<u>הערה</u>: לחשוב מרחבים סגורים יש מבנה של מבנה סימפליציאלי.

<u>הגדרה</u> $C^\Delta_*(X)$ ו- $H^\Delta_*(X)$ ותרגיל הדוגמאות

עוד שני דברים: 1. מבנה סימפליציאלי על I 2. מבנה סימפליציאלי על $|X| \times |Y|$

<u>משפט</u> $H^\Delta_*(X) = H_*(|X|)$, ויותר מזה $H^\Delta_*(X,A) = H_*(|X|,|A|)$

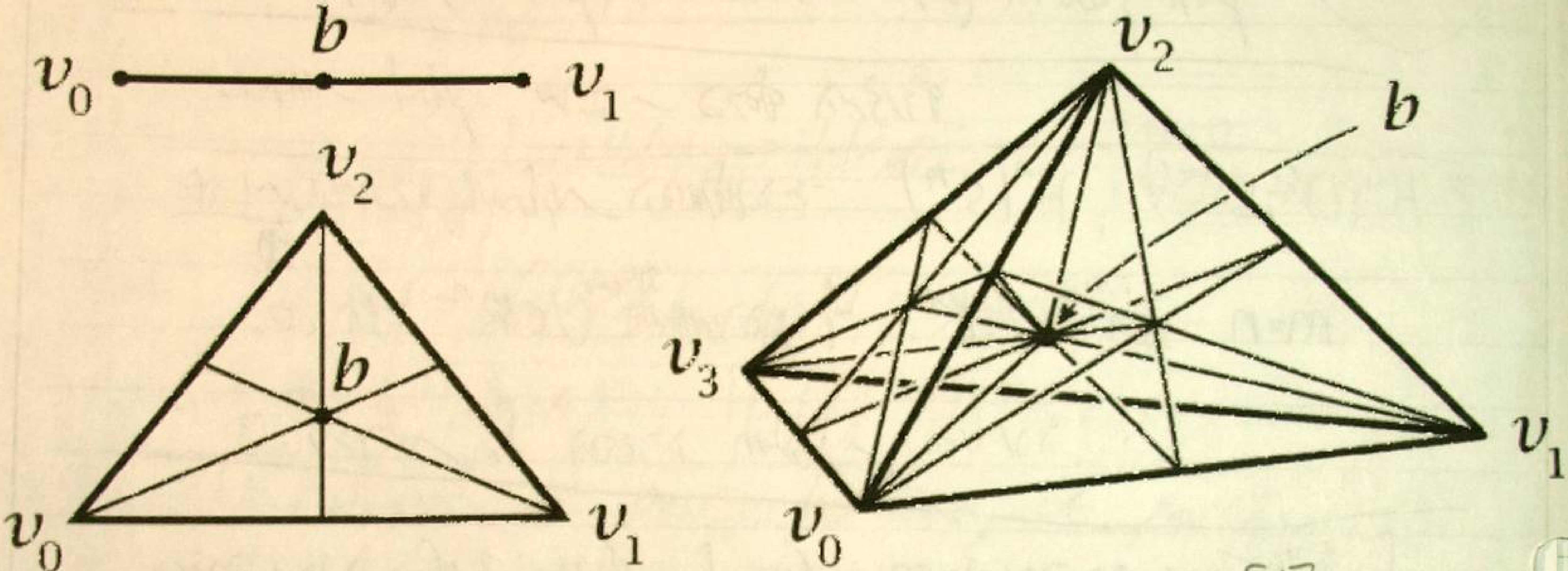
מקרים: 1. X ממימד סופי ו- A ריק.
2. X כללי ו- A ריק.
3. X, A כלליים.

טופולוגיה אלגברית – 28 לנוב' 2002 – החלוקה הבריצנטרית

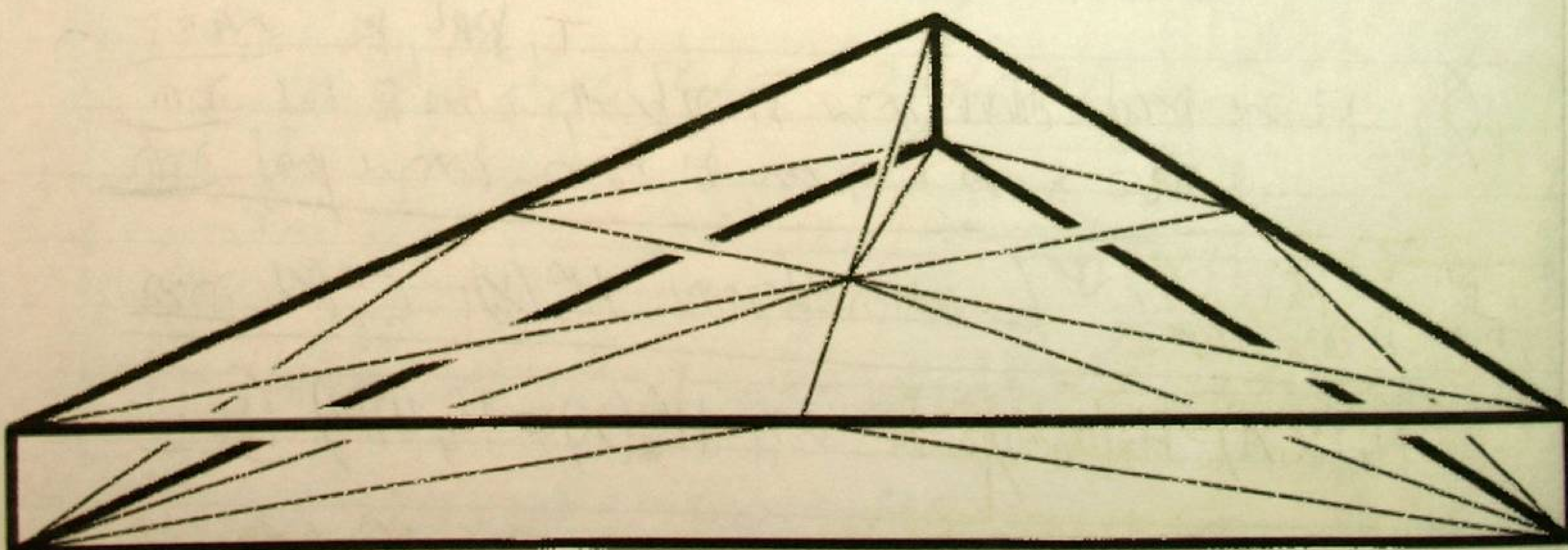
<u>הגדרות</u>: עבור $\sigma=[v_0 \dots v_n]$ נגדיר $b_\sigma=\frac{1}{n+1}\sum v_i$; ובאופן כללי,
$b[w_0 \dots w_n] := [b, w_0, \dots, w_n]$ ואינדוקטיבית, $S[\emptyset]=[\emptyset]$, $S\sigma=b_\sigma(S\partial\sigma)$

<u>טענה</u>: לכל $\sigma' \in S\sigma$, $\operatorname{diam}\sigma' \le \frac{n}{n+1}\operatorname{diam}\sigma$

<u>טענה</u>: $\partial S = S\partial$ (ולכן S מגדיר מורפיזם בין הקומפלקסים)



<u>הגדרה</u> $T\sigma = \begin{cases} 0 & \sigma=[\emptyset] \\ b_\sigma(\sigma - T\partial\sigma) & \text{אחרת} \end{cases}$



<u>טענה</u> $1-S=\partial T + T\partial$ ולכן S הומוטופית לזהות.

דניר קרימן

(Pictures from Hatcher)

Math 1300 Topology, Tue Feb 8 2005

on board.

Need $H_n(X,A) \cong \tilde{H}_n(X/A) = H(X/A, A/A)$

Excision $Z \subset A \subset X$, $\bar{A} \subset A^\circ \Rightarrow$

$$H_n(X-Z, A-Z) \xrightarrow{\sim} H_n(X,A)$$

$$\begin{array}{ccccc} H_n(X,A) & \longrightarrow & H_n(X,V) & \longleftarrow & H_n(X-A, V-A) \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{H}_n(X/A) = H_n(X/A, A/A) & \longrightarrow & H_n(X/A, V/A) & \longleftrightarrow & H_n(X/A - A/A, V/A - A/A) \end{array}$$

* Complete this discussion

* H_n^U, $H_n^U(X,A)$ Prop $H_n^U(X) \xrightarrow{\sim} H_n(X)$

$H_n^U(X,A) \xrightarrow{\sim} H_n^U(X,A)$

* Proof of excision:

$U = \{A^\circ, X \setminus Z\}$ $C_n(X-Z, A-Z) \longrightarrow C_n^U(X,A)$

* The barycentric subdivision & proof of Prop.

Continue as on May 28, 2002.

Math 1300 Topology, Thu Feb 10 2005

* Apology
* Finish excision : * $C_n^{\mathcal{U}}(X-Z, A-Z) \xrightarrow{\sim} C_n^{\mathcal{U}}(X, A)$ $\mathcal{U} = \{A^\circ, X-Z\}$
    * The barycentric subdivision
    * rest.

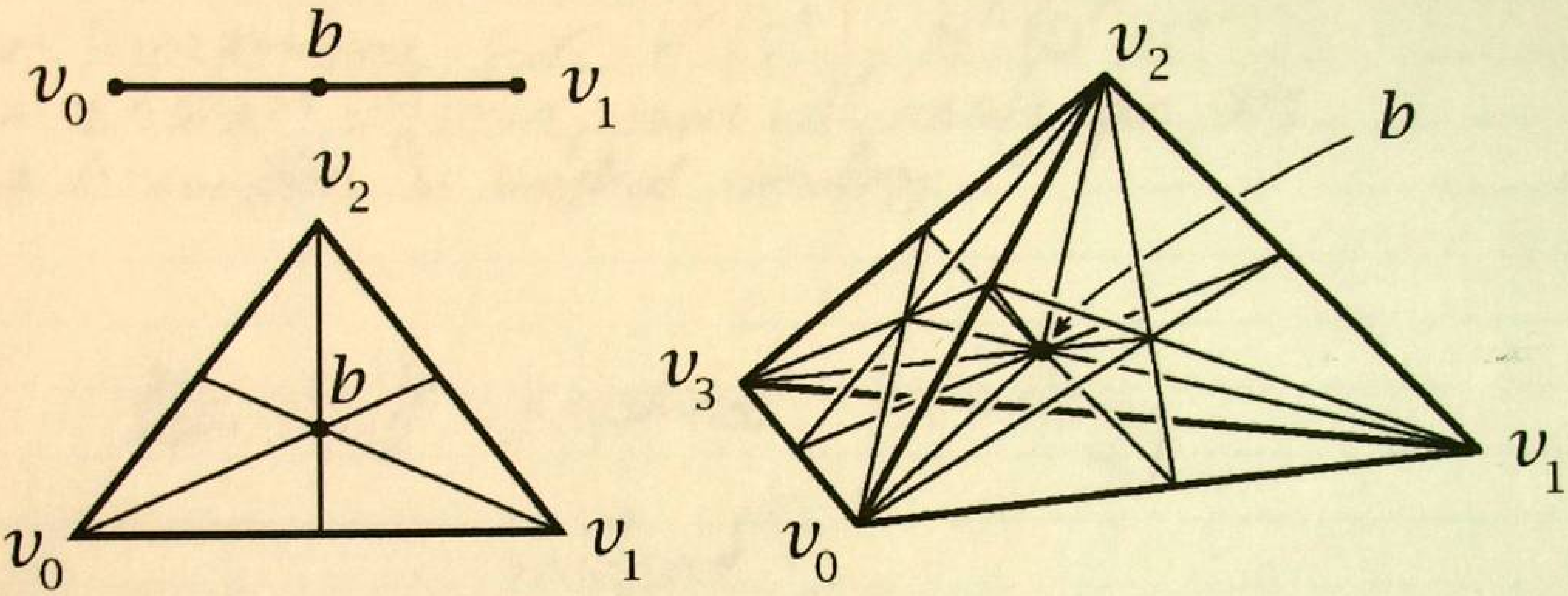
* Representatives for $H^n(S^n)$, $H^n(D^n, S^{n-1})$
* $\mathbb{R}^n \cong \mathbb{R}^m \Rightarrow m = n$, same with merely open sets.
* Δ-complexes & simplicial homology.

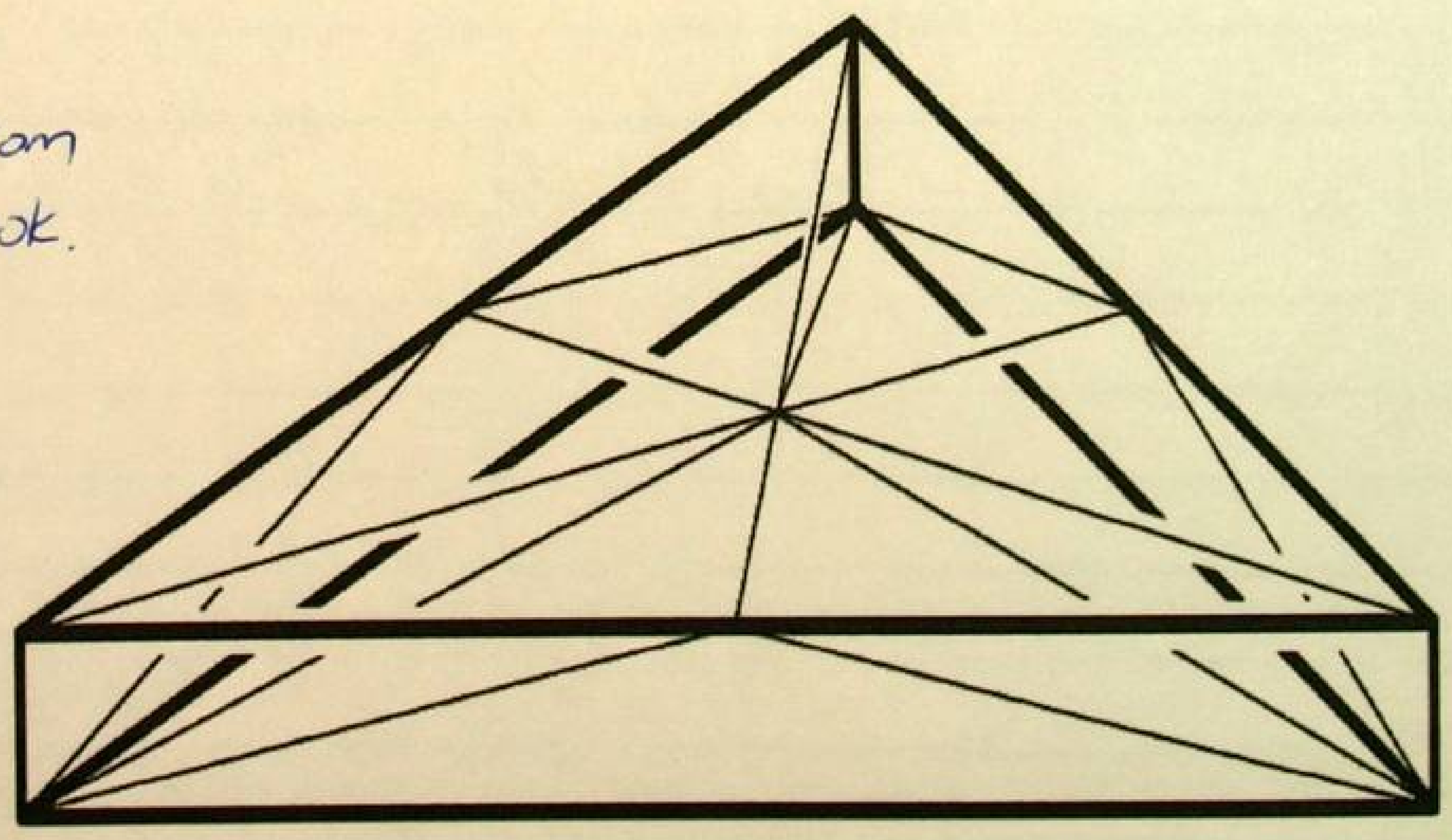
Did Khovanov homology instead!

The Barycentric Subdivision

Definition: For $\lambda=[v_0,\dots,v_n]$ set $b_\lambda=\frac{1}{n+1}\sum v_i$. In general set $b[w_0,\dots,w_n]:=[b,w_0,\dots,w_n]$ and then inductively, $S[\emptyset]=[\emptyset]$ and $S\lambda=b_\lambda(S\partial\lambda)$.

Claim For any $\sigma\in S\lambda$, $\operatorname{diam}\sigma\le\frac{n}{n+1}\operatorname{diam}\lambda$





Figures from Hatcher's book.

Claim $\partial S=S\partial$ (so S defines a map between complexes)

Definition Inductively, $T[\emptyset]=0$ and $T\lambda=b_\lambda(\lambda-T\partial\lambda)$

Claim $\partial T+T\partial=Id-S$, so S is homotopic to the identity.

Math 1300 Topology, ~~Thu~~ Tue Feb 22 2005

* Heather Lynn Sigler
* TE2 on Tue March 1 6-8 PM?

on board:

Excision $Z \subset A^\circ \Rightarrow$
$H(X,A) \cong H(X-Z, A-Z)$

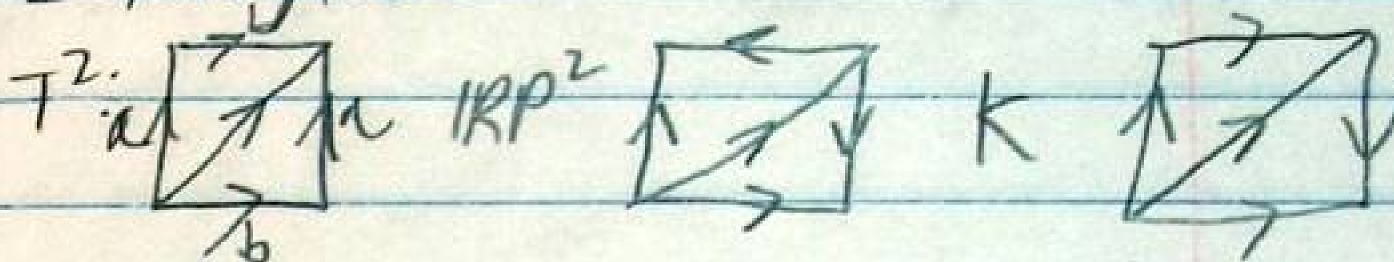
"Small" homology: $\mathcal{U}$ an open cover of X, $C_n^{\mathcal{U}} = \langle$ n-simplices σ s.t. $\sigma \in U \in \mathcal{U} \rangle$

Prop $H_n^{\mathcal{U}}(X) \longrightarrow H_n(X)$ is iso.

Excision follows:
$C_n^{\mathcal{U}}(X-Z, A-Z) \xrightarrow{\sim} C_n^{\mathcal{U}}(X,A)$
$\mathcal{U} = \{A^\circ, X-Z\}$

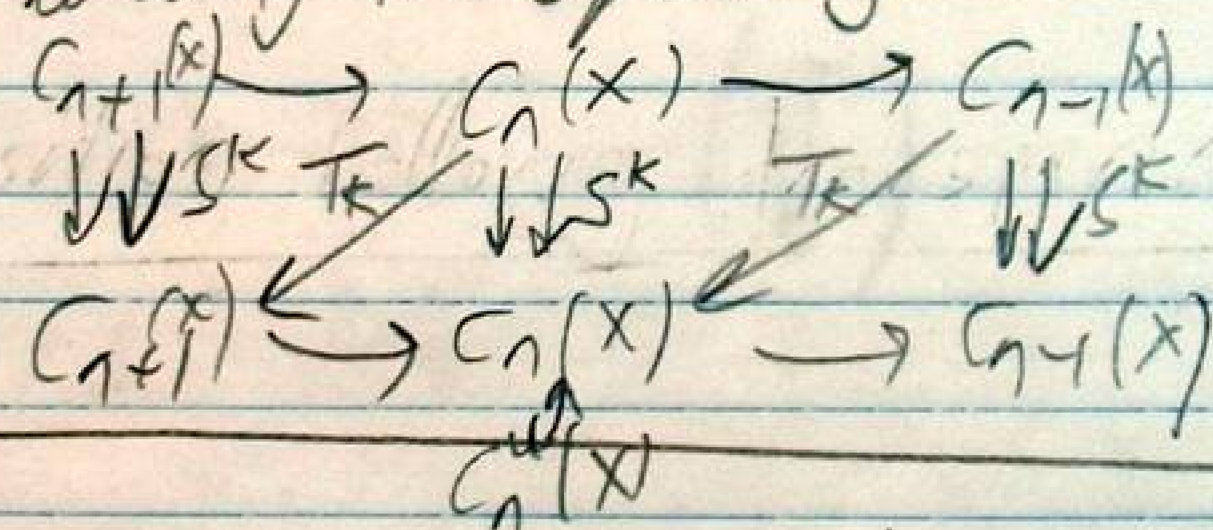
Goal: If X is made by nicely gluing simplices, it's homology can be computed just from those, and just as you would think.

Examples:



challanges: The cube I^n
Products $\Delta^n \times \Delta^m$

* Prove prop. following transparency



A Δ-complex s.t. $\partial_i \partial_j = \partial_{j-1} \partial_i$ if $i<j$

X: $S_2^X \rightrightarrows S_1^X \rightrightarrows S_0^X$ (maps $\partial_0, \partial_1, \partial_2$)

$C_n(X) = \langle S_n^X \rangle$; $\partial = \sum (-1)^i \partial_i$; $H_n(X)$

compute for T^2, $\mathbb{RP}^2$, K.

$$|X| = \coprod_{\sigma \in S_n^X} |\sigma| \Big/ \partial_i|\sigma| \leftrightarrow |\partial_i \sigma|$$

$|\sigma|$ — a simplex of dimension n

$\partial_i|\sigma|$ it's ith boundary.

Theorem $H_n^\Delta(X) \xrightarrow{\sim} H_n(|X|)$
$H_n(X,A) \xrightarrow{\sim} H_n(|X|,|A|)$

T^2, $\mathbb{RP}^2$

~/2

Feb 22 page 2

PF X_k: the kth truncation:

$$X_k: \quad S_k \Rrightarrow S_{k-1} \Rrightarrow \cdots$$

$$\begin{array}{ccccccccc} H_{n+1}(X_k, X_{k-1}) & \to & H_n(X_{k-1}) & \to & H_n(X_k) & \to & H_n(X_k, X_{k-1}) & \to & H_{n-1}(X_{k-1}) \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 & & \downarrow \alpha_4 & & \downarrow \alpha_5 \\ H_{n+1}(|X_k|, |X_{k-1}|) & \to & H_n(|X_{k-1}|) & \to & H_n(|X_k|) & \to & H_n(|X_k|, |X_{k-1}|) & \to & H_{n-1}(|X_{k-1}|) \end{array}$$

* Commutative diagram
* The five lemma.
* α_2 & α_5 are iso
* α_1 & α_4 are iso. (!)
  $\Rightarrow \alpha_3$ is iso

* The infinite case
* The relative case.

Math 1300 Topology, Thu Feb 24 2005

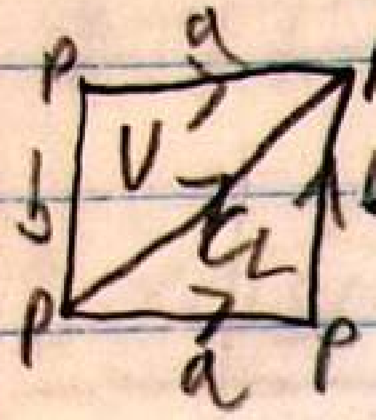
on board $S_n \overset{\partial_0}{\underset{\partial_n}{\Rrightarrow}} S_{n-1} \Rrightarrow S_{n-1}$ & full details of T.

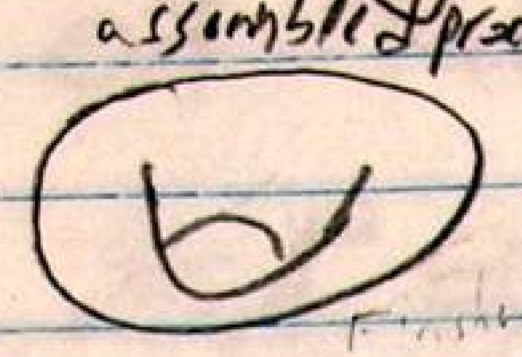
otherwise continue as Feb 22.

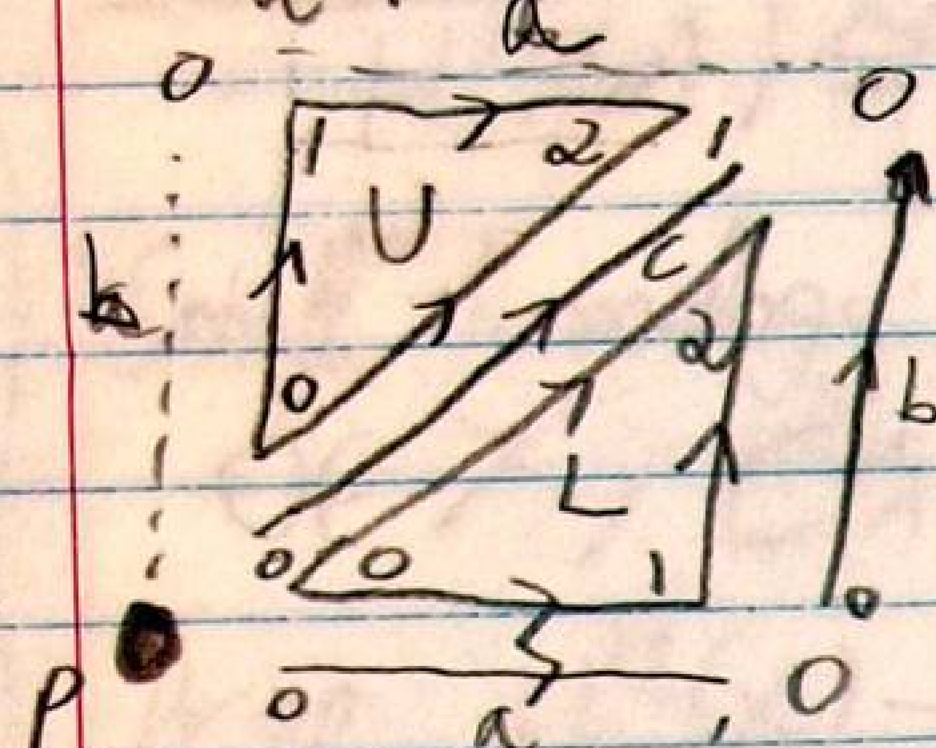
Math 1300 Topology, Thu Feb 24 2005

Torus Assemby Instructions (seen at Ikea Topoville)

on board







$$\{U,L\} \xrightarrow[\partial_2]{\partial_0} \{a,b,c\} \xrightarrow[\partial_1]{\partial_0} \{p\}$$

$$\partial_0 a = p = \partial_1 a \qquad \partial_{0,1} b = p \qquad \partial_{0,1} c = p$$

$$\partial_{0,1,2} U = (a, c, b) \qquad \partial_{0,1,2} L = (b, c, a)$$

$C.$	$\mathbb{Z}^2_{U,L} \xrightarrow[\partial_0-\partial_1+\partial_2]{\begin{pmatrix}1&1\\1&1\\-1&-1\end{pmatrix}}$	$\mathbb{Z}^3_{a,b,c} \xrightarrow[\partial_0-\partial_1]{(0\,0\,0)}$	$\mathbb{Z}^1_p$
Z	$\langle U-L\rangle$	$\langle a,b,c\rangle$	$\langle p\rangle$
B	0	$\langle a+b-c\rangle$	$\langle 0\rangle$
H	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}$

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continue as on Feb 22.

About This Class

URL: http://www.math.toronto.edu/~drorbn/classes/0405/Topology/.

Agenda: Learn about the surprising relation between the easily deformed (topology) and the most rigid (algebra).

Instructor: Dror Bar-Natan, drorbn@math.toronto.edu, Sidney Smith 5016G, 416-946-5438. Office hours: Thursdays 12:30-1:30.

Teaching Assistant: Toan Ho Minh, hmtoan@math.toronto.edu, Sidney Smith 623A, 416-978-2967.

Classes: Tuesdays 1-3 and Thursdays 2-3 at Sidney Smith 5017A.

Optimistic Plan:

1. Point set topology: Topological spaces and continuous functions, connectedness and compactness, countability and separation axioms.
2. Homotopy: Fundamental group, Van Kampen theorem, Brouwer's theorem for the 2-disk. Homotopy of spaces and maps, higher homotopy groups.
3. The language of category theory.
4. Covering theory, universal coverings.
5. Homology: Simplicial and singular homology, homotopy invariance, exact sequences, Mayer-Vietoris, excision, Brouwer's theorem for the n-disk, degrees of maps, CW-complexes, Euler characteristic, a word about the classification of surfaces.
6. Cohomology: Cohomology groups, cup products, cohomology with coefficients.
7. Topological manifolds: Orientation, fundamental class, Poincare duality.

Textbooks: We will mainly use James Munkres' Topology (ISBN 0-13-181629-2) and Allen Hatcher's Algebraic Topology (Free! ISBN 0-521-79540-0). Additional texts by Bredon, Bott-Tu, Dugundji, Fulton, Massey and others are also excellent.

Lecture Notes: I'll be happy to scan the lecture notes of one of the students after every class and post them on the web. We need a volunteer with a good handwriting!

The Final Grade: For students taking this course all year the final grade will be determined by applying an increasing continuous function (to be determined later) to $0.2HW+0.15TE1+0.15TE2+0.5F$, where HW, $TE1$, $TE2$ and F are the HomeWork, Term Exam 1, Term Exam 2 and Final grades respectively. For students taking only the second half of the course the final grade will be determined by applying an increasing continuous function (possibly a different one) to $0.2HW+0.2TE2+0.6F$.

Homework: There will be about 12 problem sets. I encourage you to discuss the homeworks with other students or even browse the web, so long as you do at least some of the thinking on your own and you write up your own solutions. The assignments will be assigned on Thursdays and each will be due on the date of the following assignment, in class at 3PM. There will be 10 points penalty for late assignments (20 points if late by more than a week and another 10 points for every week beyond that). Your 10 best assignments will count towards your homework grade. If you are only taking the second half of the course, you'll only see 7 of the assignments and only your best 6 will count towards your homework grade.

The Term Exams: Term exam 1 and Term Exam 2 will take place in the afternoons or evenings outside of class time, on the weeks of November 15 and February 28, respectively. They will be 2 hours long.

Feedback: I'd be very happy to hear from you. There's a link to a feedback form at the top of this class' web site (and here). Anonymous messages are fine, provided they are written with good intent. Though remember that if I don't know who you are I may not be able to address your concern. You will each be *required* to use this feedback form at least once, on the third week of classes (see below).

Class Photo: To help me learn your names, I will take a class photo on Thursday of the third week of classes. I will post the picture on the class' web site and you will be *required* to use the feedback form to identify yourself in the picture.

The Topology comprehensive exam syllabus, 2005

From

Topology

1. Set Theory: Zorn's lemma, axiom of choice. Point set topology: metric spaces, topological space, basis, subbasis, continuous maps, homeomorphism, subspace, quotient space, Tychonov theorem, properties of spaces (normal, Hausdorff, completely regular, paracompact), Urysohn lemma, Tietze's extension theorem, parallelotopes, Stone-Cech compactification.

   Dugundji, *Topology,* Allyn Bacon Inc., Boston, 1966, Chapters 1-7, 11-12.
   Kelley, *General Topology,* Graduate Texts in Mathematics 27, Springer-Verlag, 1955, Chapters 1-5, 7, and Appendix.

2. Algebraic topology: homotopic maps, fundamental group, Van Kampen theorem, higher homotopy groups. Covering spaces, covering transformations, map lifting, classification of covering spaces. Singular homology theory, axioms for homology. CW complexes, cellular homology, Euler characteristic. Cohomology, universal coefficient theorem, cup product. Manifolds, orientation, Poincaré duality. Applications (Brouwer fixed point theorem, Jordan-Brouwer separation theorem, invariance of domain, etc.)

   Greenberg and Harper, *Lectures on Algebraic Topology,* pp. 1-229, Second Edition, Addison-Wesley, 1981.
   Massey, *A Basic Course in Algebraic Topology,* Graduate Texts in Mathematics 127, Springer-Verlag, 1991, pp. 117-146.

* Talk about TE2.

Math 1300 Topology, Tue March 1, 2005

Finish up Δ-complexes as on Feb 22, 2005

Definition of degree: $\deg f : S^n \to S^n$

Properties

1. $\deg I = 1$
2. $\deg f = 0$ if f not onto
3. homotopy invariance
3. $\deg fg = (\deg f)(\deg g)$
4. $\deg(\text{reflection}) = -1$
5. $\deg(-I) = (-1)^{n+1}$
6. $f : S^n \to S^n$ no f.p. $\Rightarrow \deg f = (-1)^{n+1}$

Thm S^n "can be combed" iff n is odd.

The local formula.

---

on board. The commutative diagram for X_k, X_{k-1}

Math 1300 Topology, Thu March 3, 2005

on board.

$$\coprod S_k \times \Delta^k$$
$$\downarrow \pi$$
$$\sigma : \Delta^n \longrightarrow |X| = \coprod S_k \times \Delta^k / \sim$$

$y_i \in S_{k_i} \times \operatorname{int} \Delta^{k_i}$, distinct S_{k_i}

<u>Claim</u> $\sigma^{-1}(|X| - \{y_1\})$ is open.

continue with degrees as on March 1, 2005

* Exam 6-8 at 552127

Math 1300 Topology, Tue March 8 2005

* The local formula for degrees (Raymond!)

* Linking numbers.

$$H^n(S^n) \longrightarrow H^n(S^n)$$

$$H(S^n, S^n \setminus x) \longrightarrow H^n(S^n, S^n \setminus y)$$

$$H(U, U \setminus x) \xrightarrow{(f|_U)_*} H(V, V \setminus y)$$

$$\begin{array}{ccc} \| & & \| \\ \mathbb{Z} & \xrightarrow{\deg_x} & \mathbb{Z} \end{array}$$

Math 300 Topology, Thu March 10 2005

* deg reminder, linking clarification
* CW complexes & homology
* Examples
* Mayer Vietoris
* $H_1(X) = \pi_1(X)^{ab}$

*Return HW.

Math 300 Topology, Tue March 15 2005

* Exam on April 29, 2-5PM (Friday)
* Finish $\mathbb{RP}^n$
* $H_1(X) = \pi_1(X)^{ab}$
* Mayer Vietoris
* $\mathbb{R}^n$ Theorems 5&6

---

on board

$$\mathbb{RP}^n = S^n/\pm 1 = e^n \cup_f \mathbb{RP}^{n-1}, \quad f: \partial e^n = S^{n-1} \xrightarrow{\text{proj.}} \mathbb{RP}^{n-1}$$

C^{cw}: $\underset{n}{\mathbb{Z}} \longrightarrow \underset{n-1}{\mathbb{Z}} \longrightarrow \underset{n-2}{\mathbb{Z}} \to \xrightarrow{2} \underset{3}{\mathbb{Z}} \xrightarrow{0} \underset{2}{\mathbb{Z}} \xrightarrow{2} \underset{1}{\mathbb{Z}} \xrightarrow{0} \underset{0}{\mathbb{Z}}$

H^{cw}: $\mathbb{Z}/2 \quad 0 \quad \mathbb{Z}/2 \quad \mathbb{Z}$

So
$$H_k(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k=0 \\ \mathbb{Z}/2 & k<n \text{ } k \text{ odd} \\ 0 & k<n \text{ } k \text{ even} \\ \mathbb{Z} & k=n \text{ odd} \\ 0 & k=n \text{ even} \end{cases}$$

$H_k^b(X)$ for path connected X.

$\pi_1(X,b)^{ab} \cong H_1^b(X)$

Mayer Vietoris

$\mathbb{R}^n$ theorems 5&6.

Math 1300 Topology, Thu March 17 2005

* Mayer Vietoris
* $\mathbb{R}^n$ Thms 5 & 6 (Pathologies in $\mathbb{R}^n$)

Special date April 11/12 Mon/Tue for Final.

Math 1300 Topology, Tue March 22, 2005

* Exams on Thursday.

on board

M-V: $X = A^\circ \cup B^\circ \Rightarrow$ exact is

$\to H_{n+1}(X) \to H_n(A\cap B) \longrightarrow H_n(A)\oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A\cap B) \to$

Theorem 5 If D is an embedded closed k-disk in S^n, then $\tilde{H}_i(S^n \setminus D) = 0$

Theorem 6 If S is an embedded k-sphere in S^n, then $\tilde{H}_i(S^n \setminus S) = \mathbb{Z}$ for $i = n-k-1$ & 0 otherwise.

* Proof of theorem 5. (excellent exercise: Get the table of last class from this proof)

* Proof of theorem 6.
* Cor: Jordan's curve theorem & $S^{n-1} \subset S^n$
* Cor: invariance of domain

Borsuk-Ulam thm For every $g: S^n \to \mathbb{R}^n$ there's an $x \in S^n$ s.t. $g(x) = g(-x)$

Lemma $f: S^n \to S^n$ is odd $\Rightarrow$ deg f is odd.

Lemma $f: S^n \to S^n$ is odd $\Rightarrow$ let $\bar{f}: \mathbb{RP}^n \to \mathbb{RP}^n$, $\deg_{\mathbb{Z}_2} \bar{f} =$ odd.

Math 1300 Topology, Thu March 24, 2005
Prove Borsuk-Ulam

Term Exam 2

University of Toronto, March 8, 2005

Math 1300Y Students: Make sure to write "1300Y" in the course field on the exam notebook. Solve one of the two problems in part A and three of the four problems in part B. Each problem is worth 25 points. If you solve more than the required 1 in 2 and 3 in 4, indicate very clearly which problems you want graded; otherwise random ones will be left out at grading and they may be your best ones! You have an hour and 50 minutes. No outside material other than stationary is allowed.

Math 427S Students: Make sure to write "427S" in the course field on the exam notebook. Solve the four problems in part B, do not solve anything in part A. Each problem is worth 25 points. You have an hour and 50 minutes. No outside material other than stationary is allowed.

Apology: due to my travel plans grading may be slow.

Good Luck!

Part A

Problem 1. Let X be a group with product $\star$.

1. What does it mean to say that "X is a topological group"?
2. If $\gamma_1 : I \to X$ and $\gamma_2 : I \to X$ are paths in X, define $\gamma_1 \star \gamma_2 : I \to X$ by $(\gamma_1 \star \gamma_2)(t) = \gamma_1(t) \star \gamma_2(t)$. Show that $[\gamma_1 \star \gamma_2] = [\gamma_1][\gamma_2]$ in $\pi_1(X)$.
3. Show that $\pi_1(X)$ is Abelian.

Problem 2.

1. State Van-Kampen's theorem.
2. Let X be the result of identifying every edge of a hexagon with its opposite in a parallel manner (to a total of 3 edge pair identifications). Compute $\pi_1(X)$. (The hexagon comes along with its interior, but the identification occurs only on the boundary).
3. (5 points bonus) Explain in a very convincing manner how X is homeomorphic to a well known space seen in class several times.

Part B

Problem 3. Let B be a connected, locally connected and semi-locally simply connected topological space with basepoint b.

1. State the classification theorem for the category of covering spaces of B. (-4) objects only.
2. Abstractly define "the universal covering U of B" using the classification theorem.
3. Use the classification theorem to show that any connected covering X of B is covered by U.

Problem 4.

1. Define "a homotopy between two morphisms f and g of chain complexes".
2. Show that homotopy of morphism is an equivalence relation on the set of all morphisms between two given complexes.
3. Show that if $f : A \to B$ and $g : A \to B$ are homotopic morphisms of chain complexes A and B, and if $j : B \to C$ is another morphism of chain complexes, then $j \circ f$ is homotopic to $j \circ g$.

Problem 5. Let X and Y be disjoint topological spaces with basepoints x and y, respectively. Assume also that x has a neighborhood U that deformation retracts (i.e., contracts) to x and likewise that y has a neighborhood V that contracts to y. Recall that the wedge sum $X \vee Y$ is $X \cup Y / x \sim y$. What is the relationship between the homologies (reduced or not, your choice) of X, Y and $X \vee Y$? Prove your assertions. (Hint: it is a good idea to excise the "linkage point" $x \sim y$).

Problem 6. A 3-dimensional Δ-complex X is defined by

$$S_3 = \{t\} \xrightarrow{\partial_{0,1,2,3}} S_2 = \{f_0, f_1\} \xrightarrow{\partial_{0,1,2}} S_1 = \{e_0, e_1, e_2\} \xrightarrow{\partial_{0,1}} S_0 = \{v_0, v_1\},$$

with boundary maps $\partial_{0,1,2,3}t = (f_0, f_0, f_1, f_1)$, $\partial_{0,1,2}f_0 = (e_0, e_1, e_1)$, $\partial_{0,1,2}f_1 = (e_1, e_1, e_2)$, $\partial_{0,1}e_0 = (v_0, v_0)$, $\partial_{0,1}e_1 = (v_0, v_1)$ and $\partial_{0,1}e_2 = (v_1, v_1)$. $\partial t = 0$ $\partial f_0 = e_0$ $\partial f_1 = e_2$ $\partial e_0 = 0$ $\partial e_1 = v_0 - v_1$ $\partial e_2 = 0$

1. Write down the chain complex $C(X)$ (including the boundary maps).
2. Compute the homology groups $H_n(X)$ of X for $0 \le n \le 3$.
3. Can you identify $|X|$? (-?) unexplained.

Good Luck!

Math 1300 Topology, the last two weeks (2004-5)

* Finish Borsuk-Ulam
* Define cohomology
* contravariant functor
* statement of the universal coef. Thm.
* relative cohom.
* homotopy invariance, Excision
* simplicial and cellular cohom
* Mayer-Vietoris

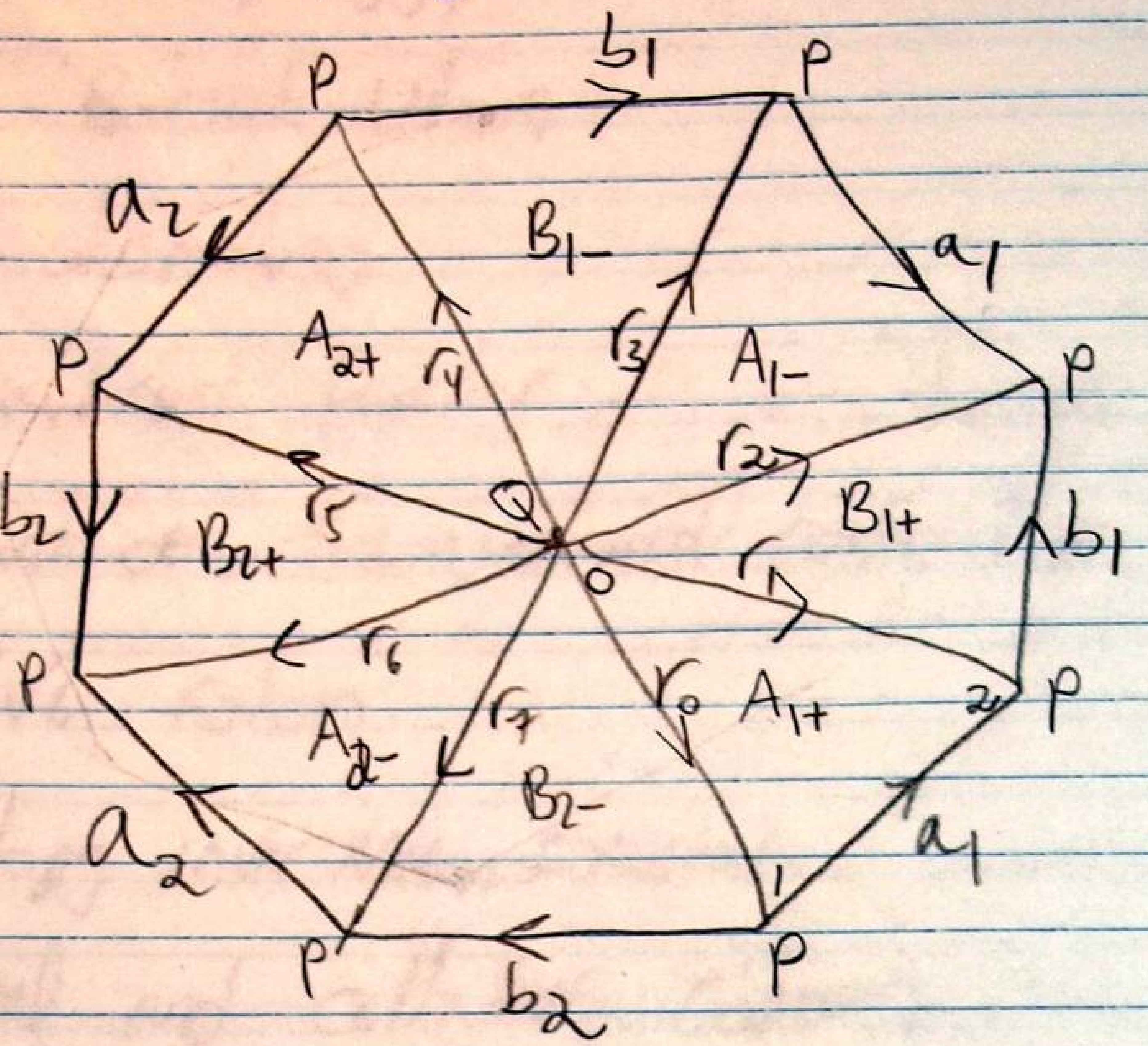
---

* Cup. def & "Lubnitz"
* Examples: surfaces
* functoriality of cup
* the torus.
* ring structure, projective space. (statement only)
* (super)-commutativity
* Kunnith (statement only?)

---

* The fundamental class
* statement of Poincare duality
* the cap product.

Math 1300 Topology – $H^*(\Sigma_2)$



Θ homology:

C_* $\quad \langle A_{i\pm}, B_{i\pm} \rangle \xrightarrow{\ \partial\ } \langle a_i, b_i, r_j \rangle \longrightarrow \langle P, Q \rangle$

$\qquad\qquad 8 \qquad\qquad\qquad 12 \qquad\qquad 2$

$\partial A_{1+} = a_1 - r_1 + r_0$ etc $\qquad \partial a_i = 0$

$\partial b_i = 0$

$\partial r_j = P - Q$

H_* $\quad \langle \sum \pm A_{i\pm} \pm B_{i\pm} \rangle \qquad \langle a_i, b_i \rangle \qquad P, Q / P = Q$

C^* $\quad \langle A^*_{i\pm}, B^*_{i\pm} \rangle \longleftarrow \langle a_i^*, b_i^*, r_j^* \rangle \xleftarrow{\ \delta\ } \langle P^*, Q^* \rangle$

$\delta A^*_{i\pm} = 0$ $\qquad \delta a_i^* = A_{i+} - A_{i-}$ $\qquad \delta P^* = \sum r_j^*$

$\delta B^*_{i\pm} = 0$ $\qquad \delta b_i^* = B_{i+} - B_{i-}$ $\qquad \delta Q^* = -\sum r_j^*$

$\delta r_0^* = A_{1+} - B_{2-}$ etc.

H^* $\quad \langle A^*_{i\pm}, B^*_{i\pm} \rangle / \text{all equiv.} \qquad \langle P^* + Q^* \rangle$

Math 1300 Topology, Tue March 29 2005

* Course Evaluation forms 15 minutes before end.
* Toan Ho Minh is having a baby!

> Today's Goal
> * Finish Borsuk-Ulam
> * Cohomology Lite
>   + all theorems are same, with arrows reversed
>   + Computations are nearly the same
> * Course evaluations

* Finish Borsuk-Ulam

* Fix G; $A \mapsto A^* := \mathrm{Hom}(A, G)$ is a contravariant functor $Ab \to Ab$.

* Define Cohomology and $\delta = \partial^*$ * interpretation of $H^{0,1}_{0,1}$ (graphs)
* reduced cohomology
* Relative Cohomology and the long exact sequence of a pair.
* functoriality
* homotopy invariance.
* Excision. $C^n_{\mathcal{U}} := (C_n^{\mathcal{U}})^* = C_n^* / (C_n^{\mathcal{U}})^{\perp}$

$$C^n \longrightarrow C^n_{\mathcal{U}} \text{ (and } S^* \text{ back) induces } H^n \xrightarrow{\sim} H^n_{\mathcal{U}}$$

> $Z \subset \mathring{A} \subset X \Rightarrow$
>
> $H^n(X - Z, A - Z) \xleftarrow{\sim} H_n(X, A)$

* simplicial cohomology / CW cohomology. } Example: $\mathbb{R}P^4$

* Mayer-Victoris: $X = \mathring{A} \cup \mathring{B} \Rightarrow$

$$H^n(X) \to H^n(A) \oplus H^n(B) \to H^n(A \cap B) \to H^{n+1}(X) \to$$

n	0	1	2	3	4
C_n	$\mathbb{Z}$ $\xleftarrow{0}$	$\mathbb{Z}$ $\xleftarrow{2}$	$\mathbb{Z}$ $\xleftarrow{0}$	$\mathbb{Z}$ $\xleftarrow{2}$	$\mathbb{Z}$
H_n	$\mathbb{Z}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0
$C_n(\mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H_n(\mathbb{Z}/2)$	same				
$C^n(\mathbb{Z})$	$\mathbb{Z}$ $\xrightarrow{2}$	$\mathbb{Z}$ $\xrightarrow{0}$	$\mathbb{Z}$ $\xrightarrow{2}$	$\mathbb{Z}$ $\xrightarrow{3}$	$\mathbb{Z}$
$H^n(\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
$C^n(\mathbb{Z}/2)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H^n(\mathbb{Z}/2)$	same.				

Universal coefficient theorem.

~~Topo~~ Math 1300 Topology, Thu March 31 2005

* Go over "dreams" handout.
* Continue as on March 29, 2005

Math 1300 Topology, Tue ~~March~~ April 5 2005.

~~* More philosophy: * The law of diminishing returns~~
~~* When the Gods of mathematics allow a structure, they want us to study it.~~
~~* Homology is there because of intersection numbers.~~

~~* Continue as on March 29, 2005~~
~~* Compute the homology ring of T^n.~~

Math 1300 Topology, Tue April 5 2005

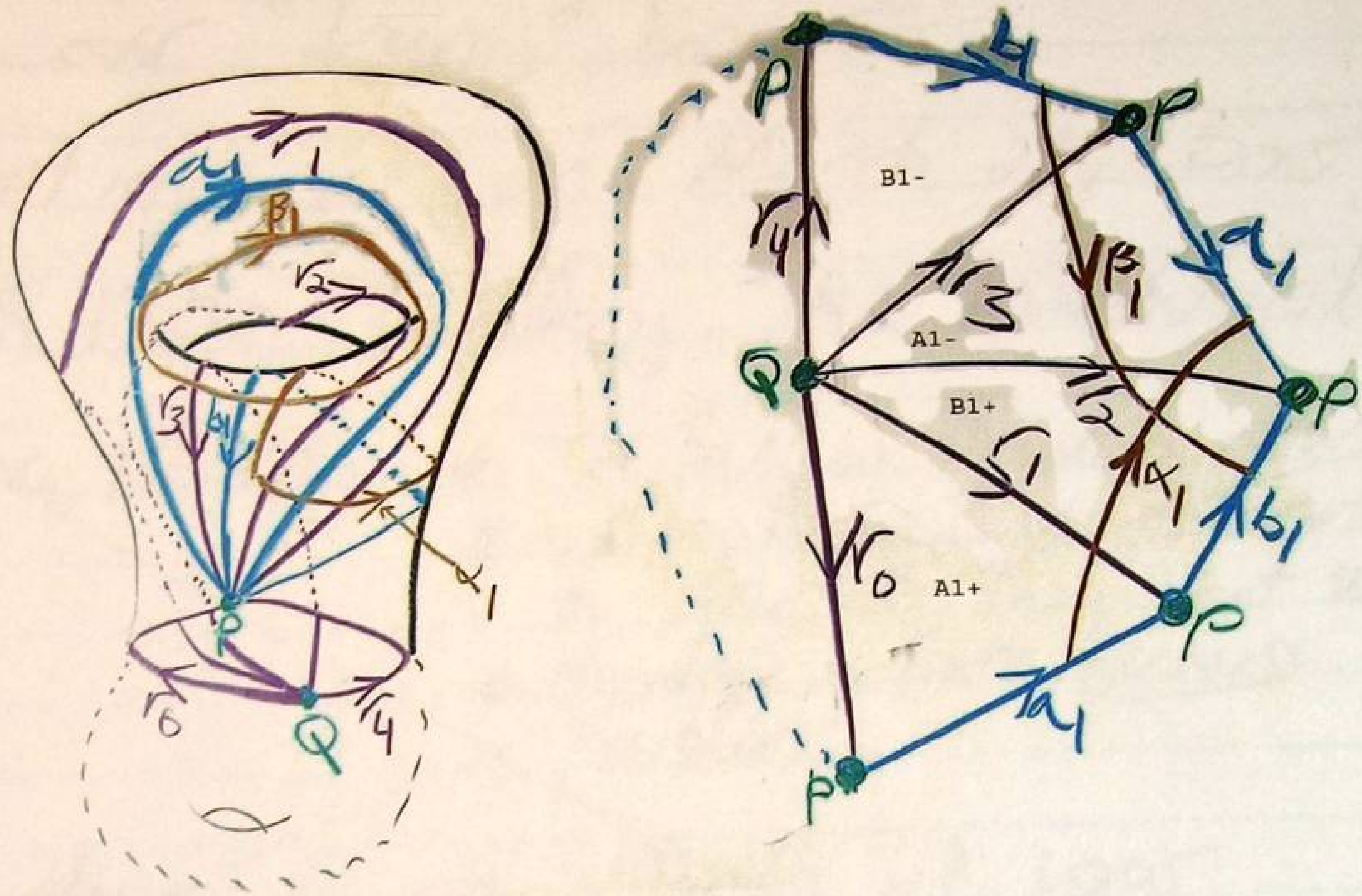
* More philosopy
  * The law of diminishing returns
  * When the Gods of Mathematics allow a structure, they want us to study it.
  * Stokes' theorem, wedge products
  * There will be applications
  * Homology is there because of intersection numbers.

* Full understanding of $H(\Sigma_g)$ (as ~~r~~ ring)
* Understanding $H^*(\mathbb{RP}^n)$ (as a v.s.)
* Continue as on March 29, 2005, with some discussion of the long exact seq & $H^n(S^n)$.
* Finish 10 minutes early for Witten.

Pre Exam Tue April 12 9-12?

A Cup Product Example

web version: http://www.math.toronto.edu/~drorbn/classes/0405/Topology/CupExample/CupExample.html



From this picture (drawn with help from Jacob Tsimerman) we can read the following:

$$\alpha_1^\dagger = a1^\star + r_1^\star + r_2^\star, \qquad -\beta_1^\dagger = b1^\star + r_3^\star + r_2^\star.$$

$$\partial_2 A_{1+} = r_0, \quad \partial_0 A_{1+} = a_1, \qquad \alpha_1^\dagger \cup \beta_1^\dagger(A_{1+}) = \alpha_1(\partial_2 A_{1+})\beta_1(\partial_0 A_{1+}) = 0\cdot 0 = 0.$$

$$\partial_2 B_{1+} = r_1, \quad \partial_0 B_{1+} = b_1, \qquad \alpha_1^\dagger \cup \beta_1^\dagger(B_{1+}) = \alpha_1(\partial_2 B_{1+})\beta_1(\partial_0 B_{1+}) = 1\cdot (-1) = -1.$$

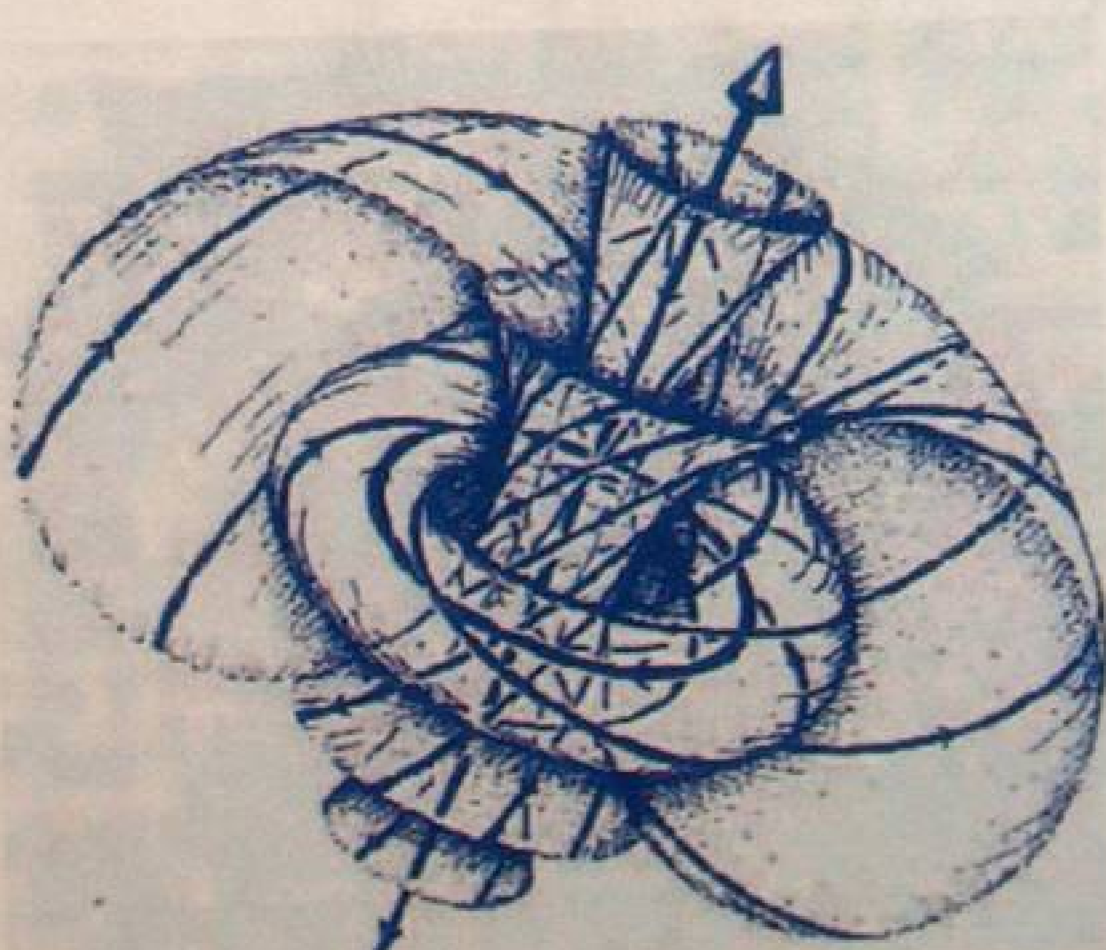
$$\partial_2 A_{1-} = r_3, \quad \partial_0 A_{1-} = a_1, \qquad \alpha_1^\dagger \cup \beta_1^\dagger(A_{1-}) = \alpha_1(\partial_2 A_{1-})\beta_1(\partial_0 A_{1-}) = 0\cdot 0 = 0.$$

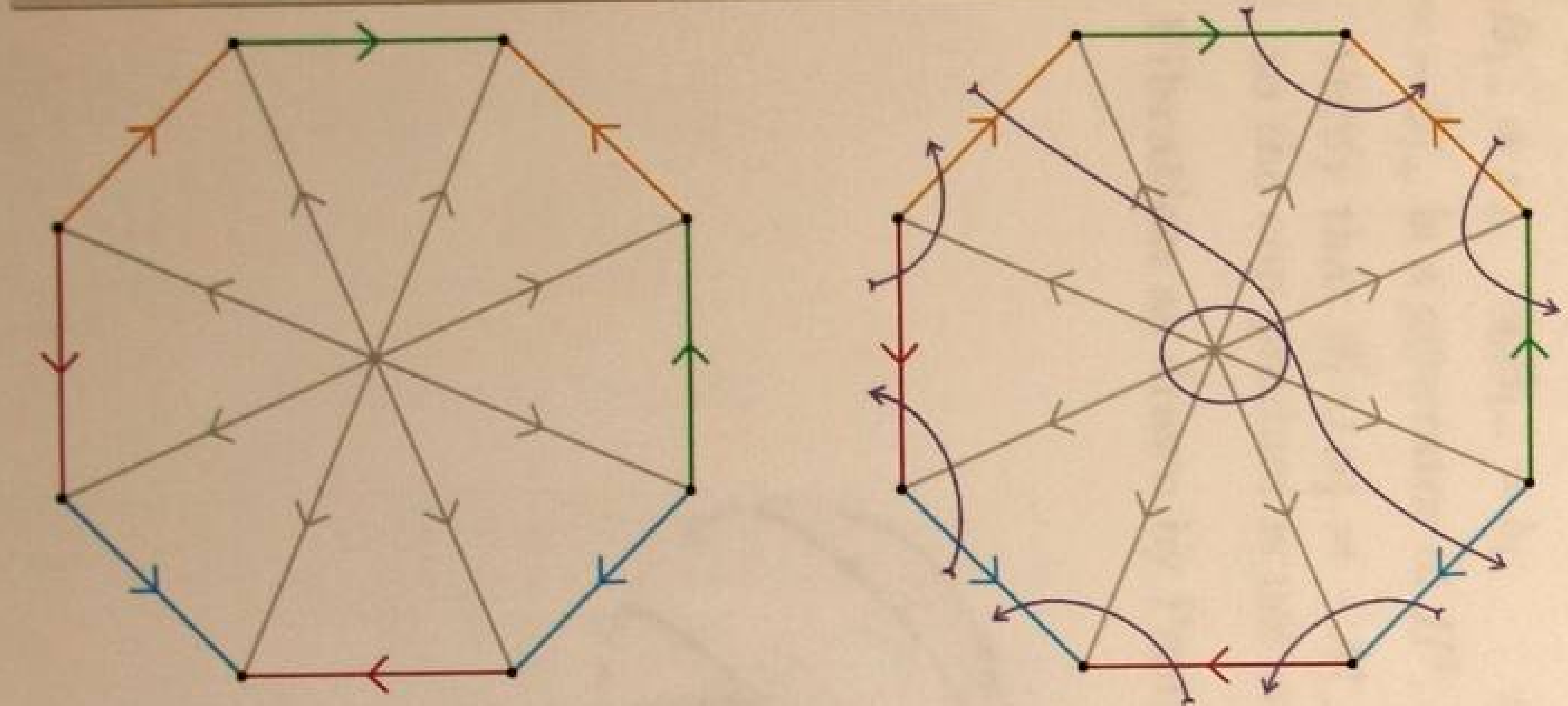
$$\partial_2 B_{1-} = r_4, \quad \partial_0 B_{1-} = b_1, \qquad \alpha_1^\dagger \cup \beta_1^\dagger(B_{1-}) = \alpha_1(\partial_2 B_{1-})\beta_1(\partial_0 B_{1-}) = 0\cdot 1 = 0.$$

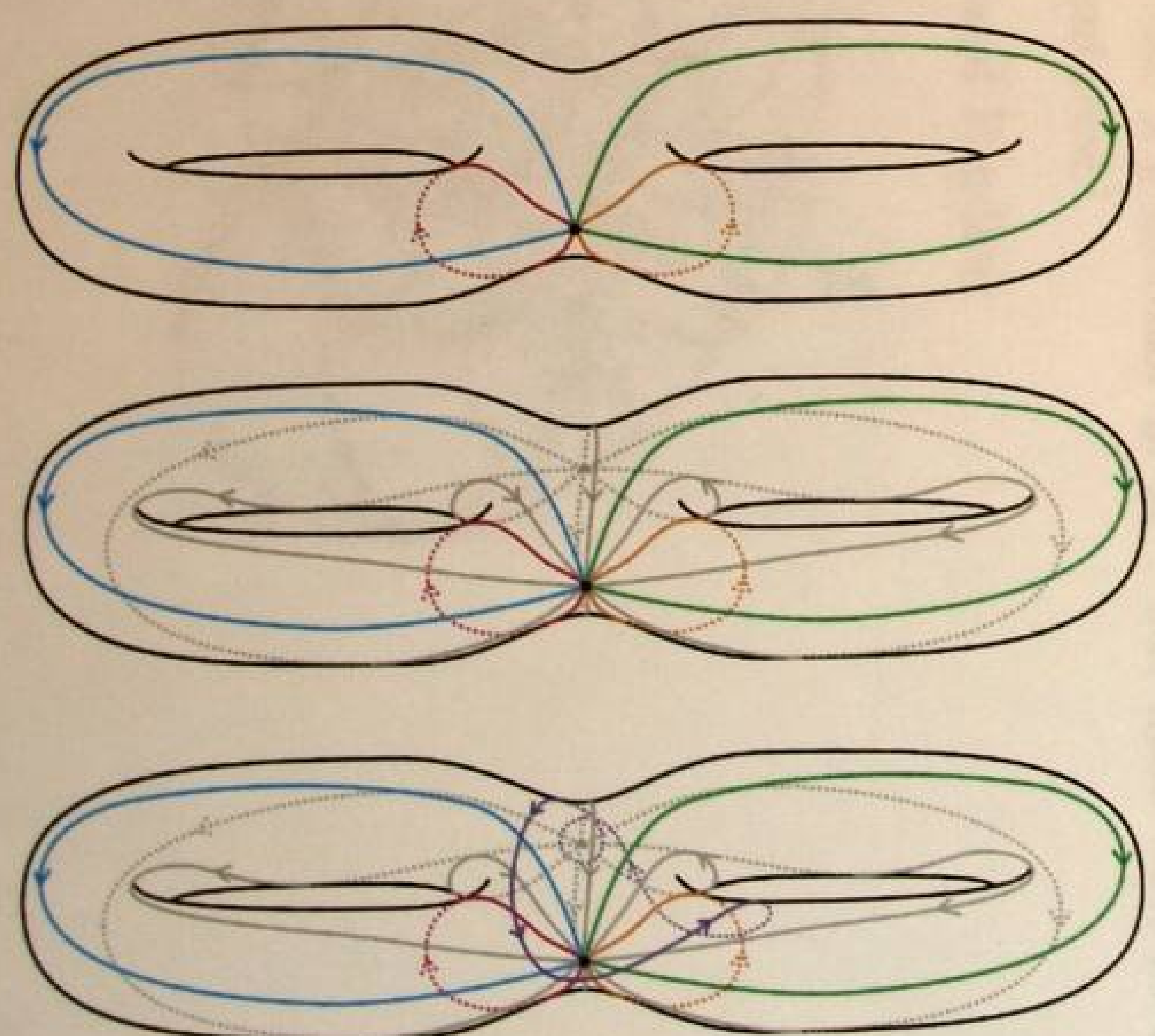
So $\alpha_1^\dagger \cup \beta_1^\dagger = -B_{1+}^\star$ is a generator of H^2.

Exercise. Verify that $\beta_1^\dagger \cup \alpha_1^\dagger = -A_{1-}^\star$ is also a generator of H^2, but note that in H^2 we have $B_{1+}^\star = -A_{1-}^\star$ so the cup product is not commutative!

The Hopf Fibration as drawn by Penrose and Rindler:







Homework Assignment 12

Assigned Thursday April 7; due Thursday April 28, 5PM, in my hand or mailbox

Required reading. Read, reread and rereread your notes to this point, and make sure that you really, really really, really really really understand everything in them. Do the same every week! Also, reread Hatcher's pages 185–217.

Solve the following problems. (But submit only the underlined ones). In Hatcher's book, problems <u>5</u>, <u>6</u>, 8, 9 on page 205 and problems <u>1</u>, 3, 11 on pages 228–229. defined on pages 6–7).

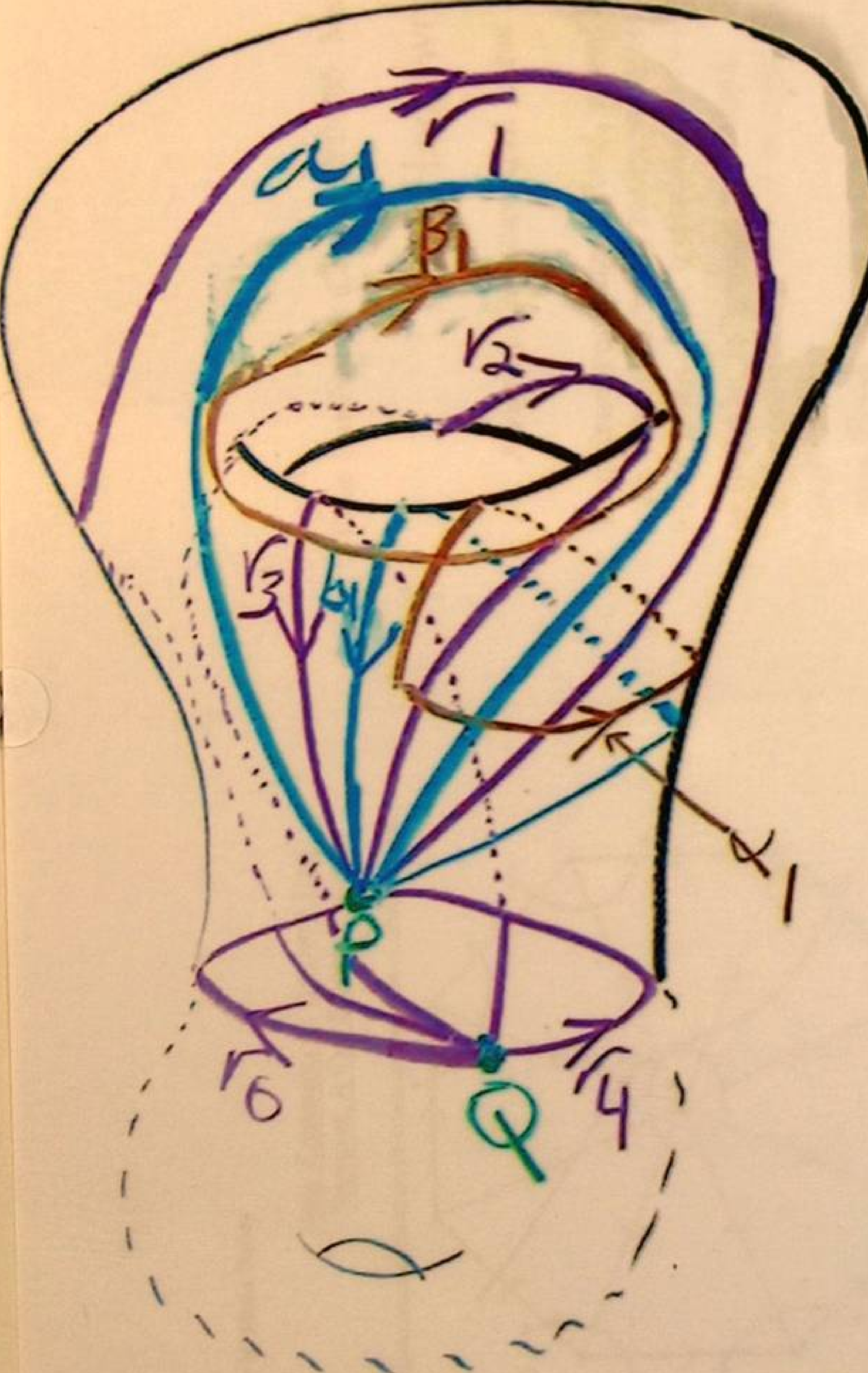
The Final Exam will take place on Friday April 29, 2-5PM, at SS 1085. On Thursday April 28 between 9–5 and on Friday April 29 between 9–12 I will be in the math lounge, or in my office with my door open, available to answer questions. If a significant group of students will hang out at the math lounge, so will I. (Though allow me a lunch break on Thursday).

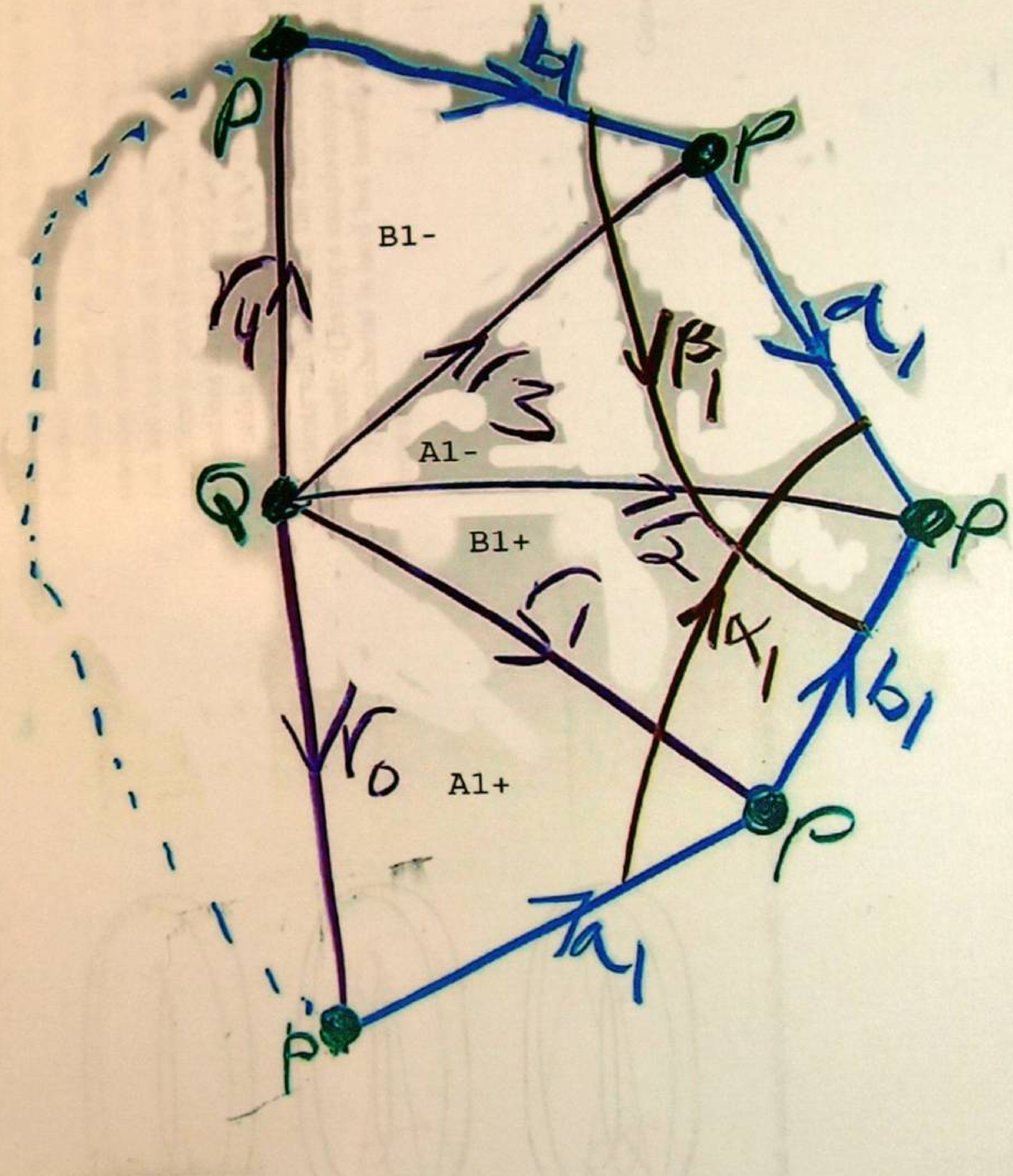
Most likely, the exam will have two parts; Part I will consist of 3 questions on first semester material (all included) and Part II will have 6 questions on second semester material (all but the class on Khovanov homology). Math 427S students will be required to solve 5 questions in part II and nothing from part I. Math 1300Y students will be required to solve 4 questions in part II and 2 questions in Part I. (So Math 1300Y will have to work harder; it is unfair but yet, they get a full year credit for the course rather than just half-year).

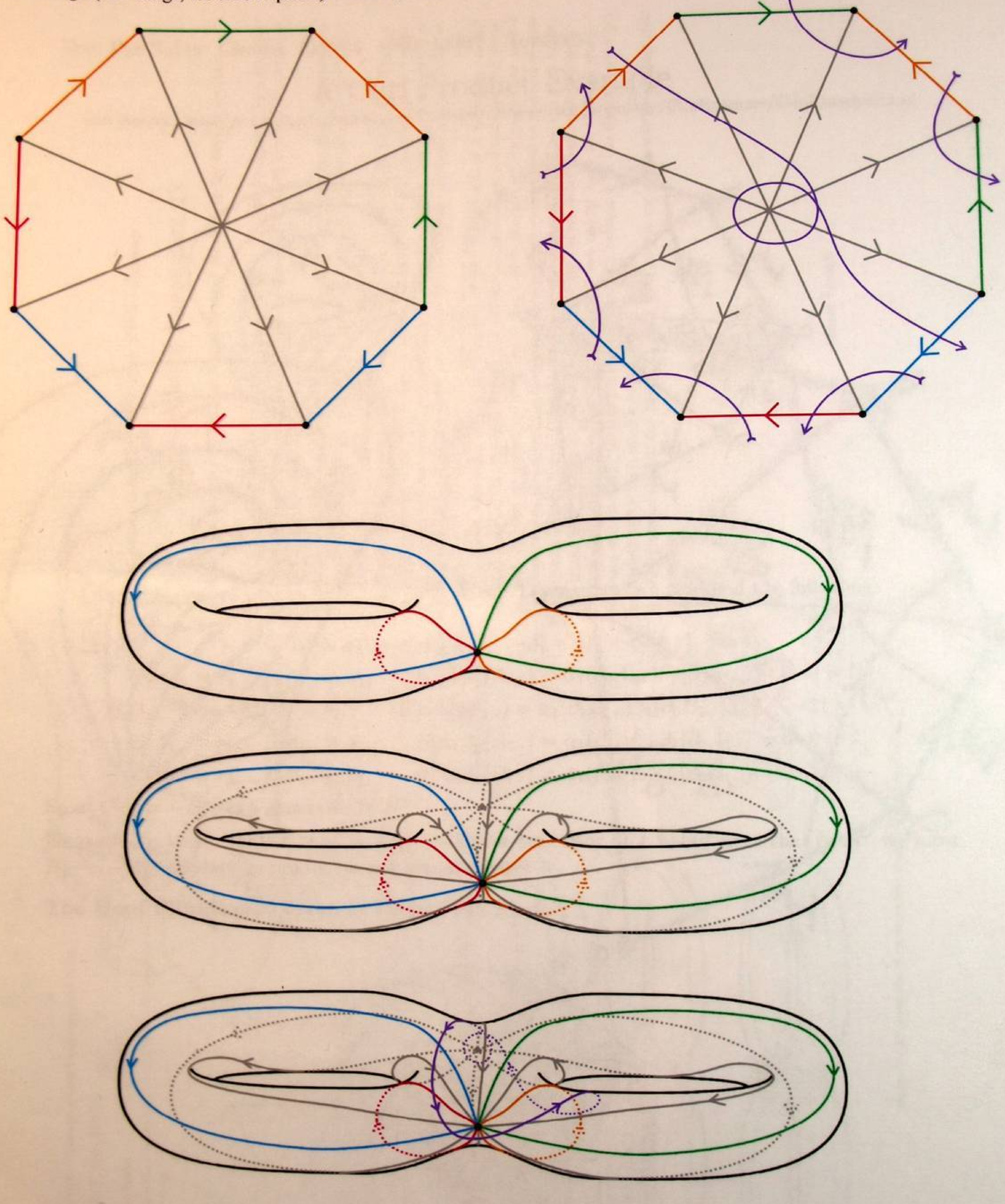
A sample exam will be available on the class' web site by April 13. (A small group of students will be taking the sample exam on April 12 between 9–12 and they will not enjoy the benefit of having a sample sample exam. Sorry.)

The material is hard but beautiful; start early so you can study with love rather than pressure.

Good Luck!!


P
Q


B1-
A1-
B1+
A1+
P
Q



Final Exam

University of Toronto, April 29, 2005

Math 1300Y Students: Make sure to write "1300Y" in the course field on the exam notebook. Solve 2 of the 3 problems in part A and 4 of the 6 problems in part B. Each problem is worth 17 points, to a maximal total grade of 102. If you solve more than the required 2 in 3 and 4 in 6, indicate very clearly which problems you want graded; otherwise random ones will be left out at grading and they may be your best ones! You have 3 hours. No outside material other than stationary is allowed.

Math 427S Students: Make sure to write "427S" in the course field on the exam notebook. Solve 5 of the 6 problems in part B, do not solve anything in part A. Each problem is worth 20 points. If you solve more than the required 5 in 6, indicate very clearly which problems you want graded; otherwise random ones will be left out at grading and they may be your best ones! You have 3 hours. No outside material other than stationary is allowed.

Good Luck!

Part A

Problem 1. Let X be a topological space.

2 1. Define the phrase "X is Hausdorff".

3 2. Define the phrase "X is normal".

2 3. Define the phrase "X is compact".

10 4. Prove that if X is compact and Hausdorff, it is normal.

Problem 2. Let X be a metric space.

3 1. Define the phrase "X is complete".

3 2. Define the phrase "X is totally bounded".

11 3. Prove that if X is totally bounded and complete than every sequence in X has a convergent subsequence.

Problem 3. (-3) "simply" connected intersection

8 1. State the Van Kampen theorem in full.

9 2. Let $D = \{z \in \mathbb{C} : |z| \leq 1\}$ be the unit disk in the complex plane and let Y be its quotient by the relation $z \sim ze^{2\pi i/3}$, for $|z| = 1$. Compute $\pi_1(Y)$.

(-6) right decomposition, wrong computation

Part B

Problem 4.

8 1. Let $p : X \to B$ be covering map and let $f : Y \to B$ be a continuous map. State in full the lifting theorem, which gives necessary and sufficient conditions for the existence and uniqueness of a lift of f to a map $\tilde{f} : Y \to X$ such that $f = p \circ \tilde{f}$. (-1) Y connected

9 2. Let $p : \mathbb{R} \to S^1$ be given by $p(t) = e^{it}$. Is it true that every map $f : \mathbb{RP}^2 \to S^1$ can be lifted to a map $\tilde{f} : \mathbb{RP}^2 \to \mathbb{R}$ such that $f = p \circ \tilde{f}$? Justify your answer.

Problem 5. Let M be an n-dimensional topological manifold (a space in which every point has a neighborhood homeomorphic to $\mathbb{R}^n$), and let p be a point in M.

7 1. Show that p has a neighborhood U for which $H_k(M - p, U - p)$ is isomorphic to $\tilde{H}_k(M)$ for all k, and so that U is homeomorphic to a ball.

5 2. Write the long exact sequence corresponding to the pair $(M - p, U - p)$.

5 3. Prove that $\tilde{H}_k(M - p)$ is isomorphic to $\tilde{H}_k(M)$ for $k < n - 1$.

(-1) No justification for $H_k(U \setminus p) = 0$ $k < n-1$.

(-3) $H_k(U - p) = 0$ for all k

Problem 6.

1. Present the space $X = S^2 \times S^4$ as a CW complex.
2. Calculate the homology of X. (I.e., calculate $H_k(X)$ for all k).
3. What is the minimal number of cells required to present X as a CW complex? Justify your answer.

Problem 7.

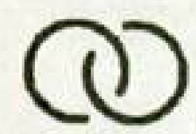
4 1. Define the *degree* $\deg \Phi$ of a continuous map $\Phi : T^2 \to S^2$.

4 2. Let $\gamma_1, \gamma_2 : S^1 \to \mathbb{R}^3$ be two continuous maps such that $\gamma_1(S^1) \cap \gamma_2(S^1) = \emptyset$. Let $\Phi_{\gamma_1,\gamma_2} : T^2 = S^1 \times S^1 \to S^2$ be defined by

$$\Phi_{\gamma_1,\gamma_2}(z_1, z_2) := \frac{\gamma_2(z_2) - \gamma_1(z_1)}{|\gamma_2(z_2) - \gamma_1(z_1)|},$$

for $z_1, z_2 \in S^1$. Prove that the degree $l(\gamma_1, \gamma_2) := \deg \Phi_{\gamma_1,\gamma_2}$ is invariant under homotopies of γ_1 and γ_2 throughout which γ_1 and γ_2 remain disjoint. (I.e., homotopies $\gamma_{1,t}$ and $\gamma_{2,t}$ for which $\gamma_{1,t}(S^1) \cap \gamma_{2,t}(S^1) = \emptyset$ for all t).

4 3. Compute (without worrying about signs, but otherwise with justification) the degree $l(\gamma_1, \gamma_2)$ where γ_1 and γ_2 are given by the picture .

5 4. Compute (without worrying about signs, but otherwise with justification) the degree $l(\gamma_1, \gamma_2)$ where γ_1 and γ_2 are given by the picture .

Problem 8.

3 1. State the theorem about the homology of the complement of an embedded disk in $\mathbb{R}^n$.

4 2. State the theorem about the homology of the complement of an embedded sphere in $\mathbb{R}^n$.

10 3. Prove that the first of these two theorems implies the second. badly based induction

Problem 9. A chain complex A is said to be "acyclic" if its homology vanishes (i.e., if it is an exact sequence). Let C be a subcomplex of some chain complex B.

1. Show that if C is acyclic then the homology of B is isomorphic to the homology of B/C (so C "doesn't matter").
2. Show that if B/C is acyclic then the homology of B is isomorphic to the homology of C (so "the part of B out of C" doesn't matter).
3. If B is acyclic, can you say anything about the relation between the homology of C and the homology of B/C?

-6 can't say nothing

Good Luck!

A Sample Final Exam

University of Toronto, April 12, 2005

Math 1300Y Students: Make sure to write "1300Y" in the course field on the exam notebook. Solve 2 of the 3 problems in part A and 4 of the 6 problems in part B. Each problem is worth 17 points, to a maximal total grade of 102. If you solve more than the required 2 in 3 and 4 in 6, indicate very clearly which problems you want graded; otherwise random ones will be left out at grading and they may be your best ones! You have 3 hours. No outside material other than stationary is allowed.

Math 427S Students: Make sure to write "427S" in the course field on the exam notebook. Solve 5 of the 6 problems in part B, do not solve anything in part A. Each problem is worth 20 points. If you solve more than the required 5 in 6, indicate very clearly which problems you want graded; otherwise random ones will be left out at grading and they may be your best ones! You have 3 hours. No outside material other than stationary is allowed.

Good Luck!

Part A

Problem 1. Let X be a topological space.

4 1. Define the "product topology" on $X \times X$.

10 2. Prove that if X is compact then so is $X \times X$.

3 3. Prove that the "folding of X along the diagonal", $S^2X := X \times X/(x,y) \sim (y,x)$ is also compact.

Problem 2. Let X be a compact metric space and let $\{U_\alpha \mid \alpha \in A\}$ be an open cover of X. Show that there exists $\epsilon > 0$ such that for every $x \in X$ there exists $\alpha \in A$ such that the ϵ-ball centred at x is contained in U_α. (ϵ is called a *Lebesgue number* for the covering.)

Problem 3.

1. Compute $\pi_1(\mathbb{RP}^2)$.
2. A topological space X_f is obtained from a topological space X by gluing to X an n-dimensional cell e^n using a continuous gluing map $f : \partial e^n = S^{n-1} \to X$, where $n \geq 3$. Prove that obvious map $\iota : \pi_1(X) \to \pi_1(X_f)$ is an isomorphism.
3. Compute $\pi_1(\mathbb{RP}^n)$ for all n.

Part B

Problem 4. Let $p : X \to B$ be a covering of a connected locally connected and semi-locally simply connected base B with basepoint b. Prove that if $p_\star\pi_1(X)$ is normal in $\pi_1(B)$ then the group of automorphisms of X acts transitively on $p^{-1}(b)$.

Problem 5. A topological space X_f is obtained from a topological space X by gluing to X an n-dimensional cell e^n using a continuous gluing map $f : \partial e^n = S^{n-1} \to X$, where $n \geq 2$. Show that

1. $H_m(X) \cong H_m(X_f)$ for $m \neq n, n-1$.
2. There is an exact sequence

$$0 \to H_n(X) \to H_n(X_f) \to H_{n-1}(S^{n-1}) \to H_{n-1}(X) \to H_{n-1}(X_f) \to 0.$$

Problem 6. Let T denote the (standard) 2-dimensional torus.

1. State the homology and cohomology of T including the ring structure. (Just state the results; no justification is required.)
2. Show that every map f from the sphere S^2 to T induces the zero map on cohomology. (Hint: cohomology flows against the direction of f).

Problem 7. For $n \geq 1$, what is the degree of the antipodal map on S^n? Give an example of a continuous map $f : S^n \to S^n$ of degree 2 (your exmple should work for every n). Explain your answers.

(-4) deg 2 only for n=1

Problem 8.

1. State the "Salad Bowl Theorem".
2. State the "Borsuk-Ulam Theorem".
3. Prove that the latter implies the former.

Problem 9. Suppose

$$\begin{array}{ccccccccc} A & \xrightarrow{a} & B & \xrightarrow{b} & C & \xrightarrow{c} & D & \xrightarrow{d} & E \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \epsilon} \\ A' & \xrightarrow{a'} & B' & \xrightarrow{b'} & C' & \xrightarrow{c'} & D' & \xrightarrow{d'} & E' \end{array}$$

is a commutative diagram of Abelian groups in which the rows are exact and α, β, δ and ϵ are isomorphisms. Prove that γ is also an isomorphism.

Good Luck!

Warning: The real exam will be similar to this sample, to my taste. Your taste may be significantly different.