Peter Lee on November 30, 2009

November-30-09
9:40 AM

A Hopf algebra A is a bialgebra with a v.s. isomorphism $S: A \to A$ s.t.

$$m(1 \otimes S)\Delta = \eta \circ \epsilon = m(S \otimes 1)\Delta.$$

[In Sweedler notation: $S(h_1)h_2 = \eta\epsilon h$]

Example: $\mathbb{Q}G$ where G is a finite group.

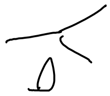
Example: $U(\mathfrak{g})$ where $\mathfrak{g}$ is a Lie algebra.

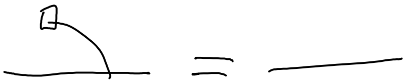
Example:

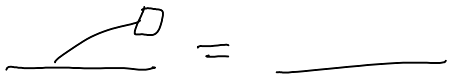
Claim $S(1) = 1$.

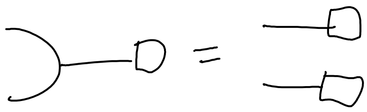
HA pictures:

η:  ϵ: 

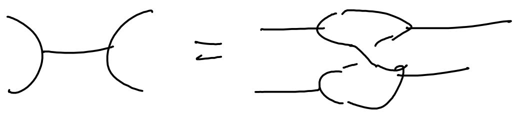


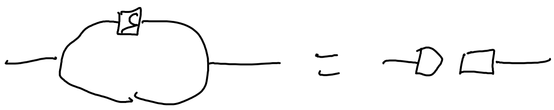
unit axiom: 

Counit axiom: 

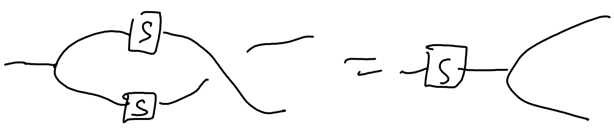
ϵ is an algebra homomorphism: 

η is a coalg homomorphism: 

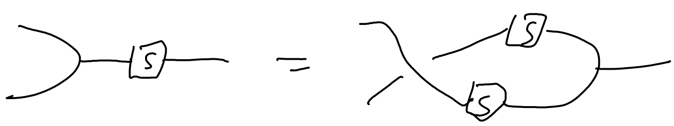
Δ/m are compatible: 

Antipode: 

Proposition



and



also  & 

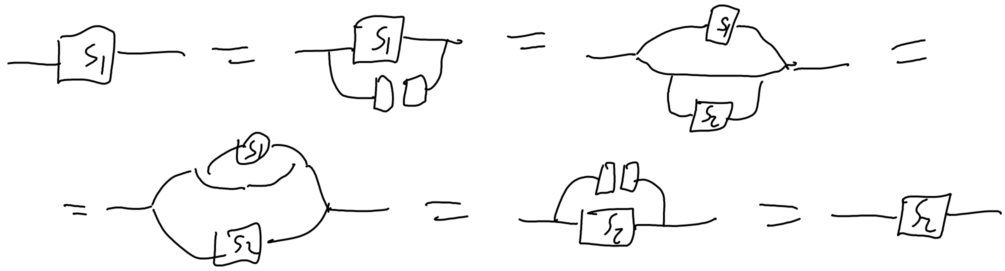
also, S is unique.

Proof of uniqueness: If S_1, S_2 are antipodes,

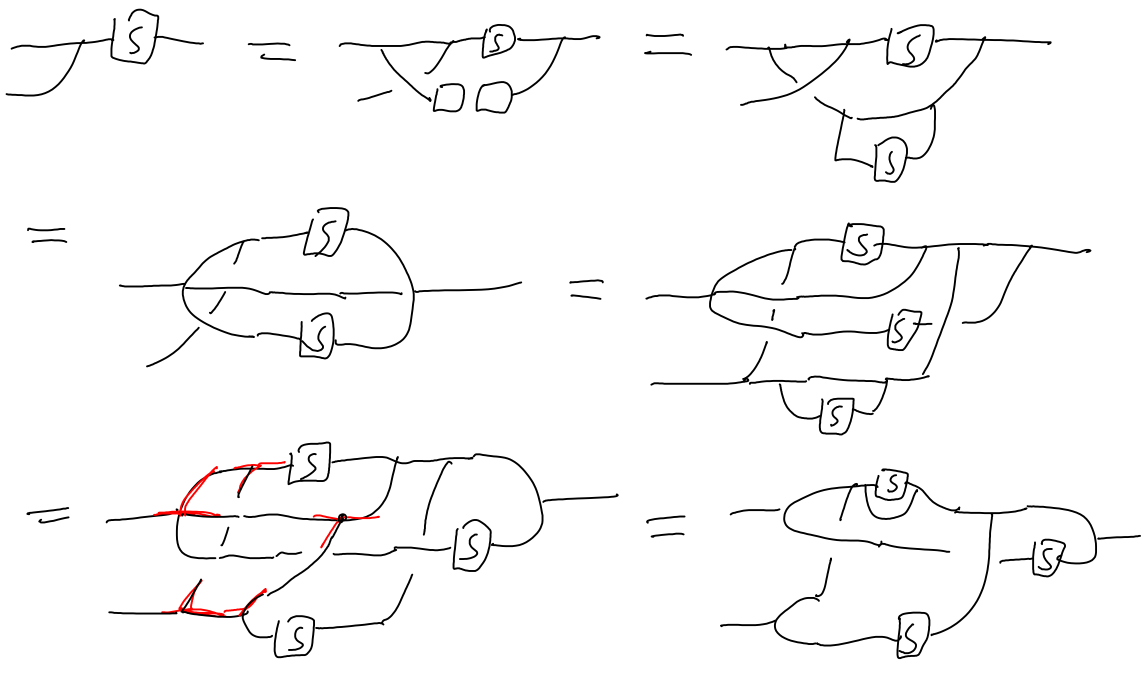
$$S_1(h) = S_1(h_1\epsilon(h_2)) = \epsilon(h_2)S_1(h_1) = (\eta\epsilon)(h_2)\, S_1(h_1)$$

$$= S_2(h_3)h_2S_1(h_1) = S_2(h_3)(\eta\epsilon)(h_2) = \dots = S_2(h)$$

In pictures:



Proof that S is an anti-isomorphism:



using  and the unit/co-unit axioms: