states that quasitriangular quantization of Section 6.1 is a functor from the category of quasitriangular Lie bialgebras to the category of quasitriangular topological Hopf algebras over $k[[h]]$. Thus, $\tau$ defines a morphism $\hat{\tau} : U_h^\text{qs}(g) \to U_h^\text{qs}(a)$. On the other hand, $U_h(a)$ was constructed as a subalgebra in $U_h^\text{qs}(g)$, so we have an embedding $\eta : U_h(a) \to U_h^\text{qs}(g)$. Consider the morphism $\tau \circ \eta : U_h(a) \to U_h^\text{qs}(a)$. This morphism is an isomorphism since it equals to 1 modulo $h$. The theorem is proved. □

Remark. An analogous theorem holds for infinite dimensional Lie bialgebras. Namely, the "usual" quantization of $\mathfrak{g}$ defined in Section 9 is isomorphic to its quasitriangular quantization. The proof is analogous to the finite dimensional case.

6.3. Representations of $U_h(g)$.

Let $A$ be a quasitriangular Lie bialgebra (not necessarily finite dimensional). By a representation of $U_h(a)$ we mean a topologically free $k[[h]]$-module $V$ together with a homomorphism $\rho : U_h(a) \to \text{End}_{k[[h]]}(V)$. Representations of $U_h(g)$ form a braided tensor category, with the trivial associativity morphism and braiding defined by the $R$-matrix. Denote this category by $\mathcal{R}$.

The functor $F : M_h \to A$ can be regarded as a functor from $\mathcal{M}_h$ to $\mathcal{R}$, since for any $W \in M_h$ the $k[[h]]$-module $F(W)$ is equipped with a natural action of $U_h(g)$. We denote this new functor also by $F$. This functor inherits the tensor structure defined by the maps $A_{ij}$.

Theorem 6.6. The functor $F$ defines an equivalence of braided tensor categories $\mathcal{M}_h \to \mathcal{R}$.

Proof. The theorem follows from the definition of the functor $F$, the algebra $U_h(g)$ and the $R$-matrix $R$. □

Part II

7. Drinfeld category for an arbitrary Lie bialgebra.

7.1. Topological vector spaces. Recall the definition of the product topology. Let $S$ be a set, $T$ a topological space, and $T^S$ the space of functions from $S$ to $T$. This space has a natural weak topology, which is the weakest of the topologies in which all the evaluation maps $T^S \to T$, $f \mapsto f(s)$, are continuous. Namely, let $B$ be a basis of the topology on $T$. For any integer $n \geq 1$, elements $s_1, \ldots, s_n \in S$, and open sets $U_1, \ldots, U_n \in B$, define $V(s_1, \ldots, s_n, U_1, \ldots, U_n) = \{ f \in T^S : f(s_i) \in U_i, i = 1, \ldots, n \}$. Let $B$ be the collection of all such sets $V$. This is a basis of a topology on $T^S$ which is called the weak topology. The obtained topological space is the product of copies of $T$ corresponding to elements of $S$.

Let $k$ be a field of characteristic zero with the discrete topology. Let $V$ be a topological vector space over $k$. The topology on $V$ is called linear if open subspaces of $V$ form a basis of neighborhoods of 0.

Remark. It is clear that in any topological vector space, an open subspace is also closed.

Let $V$ be a topological vector space over $k$ with linear topology. $V$ is called separated if the map $V \to \lim(V/U)$ is a monomorphism, where $U$ runs over open subspaces of $V$.

22

Etingof - Kazhdan Page 1
Topology on all vector spaces we consider in this paper will be linear and separated, so we will say "topological vector space" for "separated topological vector space with linear topology".

Let $M, N$ be topological vector spaces over $k$. We denote by $\text{Hom}_k(M, N)$ the space of continuous linear operators from $M$ to $N$, equipped with the weak topology. It is clear that a basis of neighborhoods of zero in $\text{Hom}_k(M, N)$ is generated by sets of the form $\{ A \in \text{Hom}_k(M, N) : Av \in U \}$, where $v \in M$, and $U \subset N$ is an open set.

In particular, if $N = k$ with the discrete topology, the space $\text{Hom}_k(M, N)$ is the space of all continuous linear functionals on $M$, which we denote by $M^*$. It is clear that a basis of neighborhoods of zero in $M^*$ consists of orthogonal complements of finite-dimensional subspaces in $M$. In particular, if $M$ is finite-dimensional, the canonical embedding $M \rightarrow (M^*)^*$ is an isomorphism of linear spaces. However, if $M$ is infinite-dimensional, this embedding is not an isomorphism of topological vector spaces since the space $(M^*)^*$ is not discrete.

7.2 Complete vector spaces

Let $V$ be a topological vector space over $k$. $V$ is called complete if the map $V \rightarrow \varprojlim(V/U)$ is an isomorphism, where $U$ runs over open subspaces of $V$.

In particular, if $M$ is a complete space $M$ has a countable basis of neighborhoods of 0, then there exists a filtration $M = M_0 \supset M_1 \supset \cdots$, such that $\bigcap_{n \geq 0} M_n = 0$, and $(M_n)$ is a basis of neighborhoods of zero in $M$. In this case $M = \lim_{n \rightarrow \infty} M/M_n$.

Examples. 1. Any discrete vector space is complete.

2. If $V$ is a discrete vector space then the topological space $M = V[[a]]$ of formal power series in $a$ with coefficients in $V$ is a complete vector space.

Let $V$ be a complete vector space, $U \subset V$ an open subspace. Then $U$ is complete and $V/U$ is discrete.

Let $V, W$ be complete vector spaces. Consider the space $V \otimes W = \varprojlim V/V_1 \otimes W/W_1$, where the projective limit is taken over open subspaces $V_1 \subset V$, $W_1 \subset W$. It is easy to see that $V \otimes W$ is a complete vector space. We call the operation $\otimes$ the completed tensor product.

A basis of neighborhoods of 0 in $V \otimes W$ is the collection of subspaces $V_1 \otimes W_1 \subset V \otimes W$, where $V_1, W_1$ are open subspaces in $V, W$.

Example. Let $V$ be a discrete space. Then $V \otimes k[[a]] = V[[a]]$.

Complete vector spaces form an additive category in which morphisms are continuous linear operators. This category, equipped with tensor product $\otimes$, is a strict symmetric tensor category.

7.3. Equicontinuous $\mathfrak{g}$-modules.

Let $M$ be a topological vector space over $k$, and $(A_x, x \in X)$ be a family of elements of $\text{End}M$. We say that the family $(A_x)$ is equicontinuous if for every neighborhood of the origin $U \subset M$ there exists another neighborhood of the origin $U' \subset M$ such that $A_x(U') \subset V$ for all $x \in X$. For example, if $M$ is complete and $A \in \text{EndM}$ is any continuous linear operator, then $(\lambda A, \lambda \in k)$ is equicontinuous.

Fix a topological Lie algebra $\mathfrak{g}$.

Definition. Let $M$ be a complete vector space. We say that $M$ is an equicontinuous $\mathfrak{g}$-module if one is given a continuous homomorphism of topological Lie algebras $\pi : \mathfrak{g} \rightarrow \text{End}M$, such that the family of operators $\pi(g), g \in \mathfrak{g}$, is equicontinuous.
Example. If $M$ is a complete vector space with a trivial $\mathfrak{g}$-module structure then $M$ is an equivariant $\mathfrak{g}$-module.

Let $V, W$ be equivariant $\mathfrak{g}$-modules. It is easy to check that $V \otimes W$ has a natural structure of an equivariant $\mathfrak{g}$-module. Moreover, $(V \otimes W) \otimes U$ is naturally identified with $V \otimes (W \otimes U)$ for any equivariant $\mathfrak{g}$-modules $V, W, U$. This means that the category of equivariant $\mathfrak{g}$-modules, where morphisms are continuous homomorphisms, is a monoidal category. This category is symmetric since the objects $V \otimes W$ and $W \otimes V$ are identified by the permutation of components. We denote this category by $\mathcal{M}_\mathfrak{g}$.

7.4. Lie bialgebras and Manin triples. Let $\mathfrak{a}$ be a Lie bialgebra over $k$. We will regard $\mathfrak{a}$ as a topological Lie algebra with the discrete topology. Let $\mathfrak{a}^*$ be the dual space to $\mathfrak{a}$. The cocommutator defines a Lie bracket on $\mathfrak{a}^*$ which is continuous in the weak topology, so $\mathfrak{a}^*$ has a natural structure of a topological Lie algebra.

Furthermore, the space $\mathfrak{a} \otimes \mathfrak{a}^*$ has a natural topology, and the Lie bracket on $\mathfrak{g}$ is defined by (1.1) is continuous in this topology.

Let $\mathfrak{g}$ be a Lie algebra with a nondegenerate invariant inner product $\langle \cdot, \cdot \rangle$. So far we have no topology on $\mathfrak{g}$. Let $\mathfrak{g}_+, \mathfrak{g}_-$ be isotropic Lie subalgebras in $\mathfrak{g}$, such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. We define an embedding $\mathfrak{g}_- \rightarrow \mathfrak{g}_+^*$. If this embedding is an isomorphism then we equip $\mathfrak{g}$ with a topology, by putting the discrete topology on $\mathfrak{g}_+$ and the weak topology on $\mathfrak{g}_-$. If in addition the cocommutator in $\mathfrak{g}$ is continuous in this topology then the triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is called a Manin triple.

To every Lie algebra $\mathfrak{g}$ one can associate the corresponding Manin triple $(\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^*, \mathfrak{a}, \mathfrak{a}^*)$, where the Lie structure on $\mathfrak{g}$ is as above. Conversely, if $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple then $\mathfrak{g}_+$ is naturally a Lie bialgebra: the pairing $\langle \cdot, \cdot \rangle$ identifies $\mathfrak{g}_+$ with $\mathfrak{g}_-$, which defines a cocommutator on $\mathfrak{g}_+^*$. This cocommutator turns out to be dual to a 1-cocycle (cf. [Dr1]).

Thus, there is a one-to-one correspondence between Lie bialgebras and Manin triples.

Let $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ be a Manin triple. Let $\{a_i \in I\}$ be a basis of $\mathfrak{g}_+$, and $b^i \in \mathfrak{g}_-$ be the linear functions on $\mathfrak{a}$ defined by $b^i(a_j) = \delta_{ij}$. We denote the linear form $b^i$ on $\mathfrak{a}$ by $\delta^i_j$.

Lemma 7.1. Let $\mathcal{M}$ be an equivariant $\mathfrak{g}$-module. Then for any $v \in M$ and any neighborhood of zero $U \subset M$ one has $b^i v \in U$ for all but finitely many $i \in I$.

Proof. We assume that $\dim \mathfrak{g}_+ = \infty$ (otherwise there is nothing to prove).

Let $\{v_m \in I : m \geq 1\}$ be any sequence of distinct elements. The $\delta^i_j v_m \to 0$, $m \to \infty$, so $b^i v_m \to 0$, $m \to \infty$, for any $v \in M$. This means that $b^i v \in U$ for almost all $i$. \( \Box \)

7.5. Examples of equivariant $\mathfrak{g}$-modules.

In this section we will construct examples of equivariant $\mathfrak{g}$-modules in the case when $\mathfrak{g}$ belongs to a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$. Consider the Verma modules $M_\alpha = \text{Ind}_{\mathfrak{g}^-}^{\mathfrak{g}^+} \mathfrak{g}_+ 1$, $M_\alpha = \text{Ind}_{\mathfrak{g}^-}^{\mathfrak{g}^+} 1$, (here $1$ denotes the trivial 1-dimensional representation). The modules $M_\alpha$ are freely generated over $U(\mathfrak{g}_+)$ by a vector $1_\alpha$ such that $\mathfrak{g}_+ 1_\alpha = 0$, and thus are identified (as vector spaces) with $U(\mathfrak{g}_+)$ via $1_\alpha \mapsto 1$.

Below we show that the module $M_\alpha$ and the module $M_\alpha^*$ dual to $M_\alpha$ in an appropriate sense are equivariant $\mathfrak{g}$-modules.
Lemma 7.2. The module $M_\varphi$ equipped with the discrete topology, is an equicontinuous $\mathfrak{g}$-module.

Proof. In order to prove the continuity of $\pi_{M_\varphi}(g)$ as a function on $\mathfrak{g}$, we have to check that for any $v \in M_\varphi$, the space $g.v \subset M_\varphi$ is finite dimensional. One may assume that $v = a_1, a_2, \ldots, a_n$. We show that $g.v$ is finite dimensional by induction in $n$. The base of induction is clear since $g.v = 0$ if $n = 0$. Now assume that $v = a_{n+1}$, where $w = a_1, a_2, \ldots, a_n$. By the induction assumption, we know that $g.w$ is finite dimensional. For any $b \in \mathfrak{g}$, we have $bw = [b, v] = [b, a_{n+1}].$ For any $j \in I$ we denote by $W_j \subset \mathfrak{g}$ the space of all $h \in \mathfrak{g}$ such that $(1 \otimes b)(\delta_{(a_j)}) = 0$. It is clear that $W_j$ has finite codimension. For any $h \in W_j$, we have $[b, h] \in \mathfrak{g}$, since $\text{ad}^\mathfrak{g}(a_j) = 0$ by the definition of $W_j$. Therefore, for any $b \in W_j$ we have $[b, v] = [b, a_{n+1}]$. Hence, $W_j \subset g.v$. The latter space is finite dimensional, which implies that $W_j \subset g.v$ is finite dimensional. Since $W_j$ has a finite codimension in $\mathfrak{g}$, the space $g.v$ is finite dimensional. This implies the continuity of the homomorphism $\pi_{M_\varphi} : \mathfrak{g} \to \text{End} M_\varphi$.

The equicontinuity condition is trivial. $\square$

Let us now introduce a topology on the space $M_\varphi$. This topology comes from the identification of $M_\varphi$ with $U(\mathfrak{g})$. The space $U(\mathfrak{g})$ can be represented as a union of $U_n(\mathfrak{g})$, $n \geq 0$, where $U_n(\mathfrak{g})$ is the set of all elements of $U(\mathfrak{g})$ of degree $\leq n$. Furthermore, for any $n \geq 0$, we have a linear map $U_n(\mathfrak{g}) \to U_n(\mathfrak{g})$ given by $x_1 \otimes \ldots \otimes x_n \mapsto x_1 \ldots x_n$. This map induces a linear isomorphism $\xi_n : \otimes^\oplus U_n(\mathfrak{g}) \to U_n(\mathfrak{g})$, where $\otimes^\oplus U_n(\mathfrak{g})$ is the $n$-th symmetric power of $\mathfrak{g}$ (as usual we set $\otimes^0 \mathfrak{g} = \mathfrak{g}$). Since $\otimes^\oplus U_n(\mathfrak{g})$ has a natural weak topology, coming from its embedding to $(U^\oplus)^n$, the isomorphism $\xi_n$ defines a topology on $U_n(\mathfrak{g})$. Moreover, by the definition, if $m < n$ then $U_m(\mathfrak{g})$ is a closed subspace in $U_n(\mathfrak{g})$. This allows us to equip $U(\mathfrak{g})$, i.e. $M_\varphi$, with the topology of inductive limit. By the definition, a set $U \subset U(\mathfrak{g})$ is open in this topology if and only if $U \cap U_n(\mathfrak{g})$ is open for all $n$.

Lemma 7.3. Let $g \in \mathfrak{g}$. Then $\pi_{M_\varphi}(g)$ is a continuous operator $M_\varphi \to M_\varphi$.

Proof. Let $g \in \mathfrak{g}$. We need to show that for any neighborhood of the origin $U \subset M_\varphi$, there exists a neighborhood of the origin $U' \subset M_\varphi$ such that $\pi_{M_\varphi}(g)(U') \subset U$.

Let $U \subset U(\mathfrak{g})$ be a neighborhood of zero, and $U_n = U \cap U_n(\mathfrak{g})$. To construct $U'$, we need to construct $U'_n = U' \cap U_n(\mathfrak{g})$ such that $U'_n = U'_n \cap U_n(\mathfrak{g})$. Before giving the construction of $U'_n$, we make some definitions.

For any neighborhood $U$ of zero, there exists an increasing sequence of finite subsets $T_n \subset I$, $n \geq 1$, such that for any $f \in S^n \mathfrak{g}$, $m \leq n$ satisfying the equation $f(a_1, \ldots, a_m) = 0$ for any $1, \ldots, m \in T_n$, one has $\xi_n(f) \in U$. Fix such a sequence $\{T_n, n \geq 1\}$.

Let $I$ be as in Section 7.4. For any finite subset $J \subset I$ denote by $S(J)$ the set of all $f$ such that there exists $b \in \mathfrak{g}$ and $j \in J$ with the property $b/\mathfrak{g}(a_j) \neq 0$. Since $[b/\mathfrak{g}(a_j)] = b \otimes b(\delta_{(a_j)})$, the set $S(J)$ is finite. Let the sets $S_n(J) \subset I$ be defined recursively by $S_0(J) = I$, $S_n(J) = S(S_{n-1}(J))$.

To construct $U'$, we consider separately the cases $g \in \mathfrak{g}_+$ and $g \in \mathfrak{g}_-$. First consider the case $g \in \mathfrak{g}_+$.

For any elements $x_1, \ldots, x_n \in \mathfrak{g}_+$, $\{n \geq 1\}$ consider the element $X = \sum x_{a_1} \otimes x_{a_2} \otimes \ldots \otimes x_{a_n}$ in $U_n(\mathfrak{g})$, where $S_n$ is the symmetric group. Consider the element $gX \in U_{n+1}(\mathfrak{g})$. It is easy to see that it is possible to write $gX$ as a linear combination of elements of the form $\sum_{x \in \mathfrak{g}_+} y_x(x)(1 - b(\delta_{(a_n)}))$, $y_x \in \mathfrak{g}_+$, $0 \leq m \leq n + 1$, in such a way that $y_x$ are
iterated commutators of $g$ and $x_1, \ldots, x_n$, and the number of commutators involved in each term $g_i$ does not exceed $n$.

Now we make a crucial observation.

Claim. Let $J \subseteq I$ be a finite subset. If for some $m$, $1 \leq m \leq n$, we have $x_{a_i}(a_i) = 0$, for all $i \in S_m(J)$, then every monomial $g_{a_1} \cdots g_{a_n}$ in the symmetrized expression of $g X$ contains a factor $g_i$ such that $g_i(a_i) = 0$, $i \in J$.

Proof. Clear.

The construction of $U'$ is as follows. For $n \geq 1$, let $U'_n \in U_n(g_{-})$ be the span of all elements $\zeta_n(f), f \in S^n g_-$, such that $f(a_{i_1}, \ldots, a_{i_n}) = 0$ whenever $i_1, \ldots, i_n \in S_{n}(T_{n+1})$. Also, set $U'_0 = 0$ (recall that $\emptyset \in k$ is a neighborhood of zero since $k$ is discrete). Our observation shows that for any $X \in U_n(g_{-})$, $g \in U_{n+1}$, as desired.

Now consider the case $g \in g_{-}$.

Let $R(g) \subseteq I$ be the set of all $i \in I$ such that $R_i(g) \neq 0$. This is a finite set. Define inductively the sets $R_n(g)$ by $R_1(g) = S(R_{n-1}(g))$.

For any finite subsets $K, J \subseteq I$ such that there exists $j \in J$ and $k \in K$ with $[a_j b_k](a_j) \neq 0$. It is clear that if $K, J$ are finite then $P(K, J)$ is finite. Let $P_n(K, J)$ be defined inductively by $P_0(K, J) = P(K, P_{n-1}(K, J))$.

Let $n \geq 1$ be an integer, $X \in U_n(g_{-})$ be as above, and $K = R_n(g)$. Consider the vector $g X 1_{K} \in M_{K}$. Using the relations in $M_{K}$, we can reduce this vector to a linear combination of vectors of the form $\sum_{i \in S_{m}} y_{1i} \cdots y_{mn}, y_{i} \in g_{-}, 0 \leq m \leq n+1$, in such a way that $g_i$ are obtained by iterated commutation of $g, x_1, \ldots, x_n$. As before, it is easy to see that the resulting symmetrized expression will contain no more than $n$ commutators.

Now let us make a crucial observation.

Claim. Let $J \subseteq I$ be any finite subset. If for some $m$, $1 \leq m \leq n$, we have $x_{a_i}(a_i) = 0$, for all $i \in S_{n}(P(K, S_{m}(J)))$, then every monomial $g_{a_1} \cdots g_{a_n}$ in the symmetrized expression of $g X 1_{K}$ contains a factor $g_i$ such that $g_i(a_i) = 0$, $i \in J$.

Proof. Clear.

The construction of $U'$ is as follows. For $n \geq 1$, let $U'_n \in U_n(g_{-})$ be the span of all elements $\zeta_n(f), f \in S^n g_-$, such that $f(a_{i_1}, \ldots, a_{i_n}) = 0$ whenever $i_1, \ldots, i_n \in S_{n}(P(K, S_{m}(T_{n+1})))$. Also, set $U'_0 = 0$. Our observation shows that for any $X \in U_n(g_{-}), g X \in U_{n+1}$, as desired.\]

Consider the vector space $M_{K}^*$ of continuous linear functionals on $M_{K}$. By the definition, $M_{K}^*$ is naturally isomorphic to the projective limit of $U_n(g_{-})^*$ as $n \to \infty$. As vector spaces, $U_n(g_{-})^* = (S^n g_{-})^* = S^n g_{-}$. Therefore, it is natural to put the discrete topology on $U_n(g_{-})^*$. This equips the module $M_{K}^*$ with a natural structure of a complete vector space. It is also equipped with a filtration by subspaces $(M_{K}^*)_n = U_{n-1}(g_{-})^*$, $n \geq 1$, such that $M_{K} = \lim_{\to} (M_{K}^*)_n$.

Remark. The topology of projective limit on $M_{K}^*$ does not, in general, coincide with the weak topology of the dual. In fact, it is stronger than the weak topology.

By Lemma 7.3, $M_{K}^*$ has a natural structure of a $g$-module. Namely, the action of $g$ on $M_{K}^*$ is defined to be the dual to the action of $g$ on $M_{K}$.

Lemma 7.4. $M_{K}^*$ is an equicontinuous $g$-module.

Proof. It is easy to see that $a(M_{K}^*)_n \subseteq (M_{K}^*)_n, a \in g_{+}$, and $b(M_{K}^*)_n \subseteq (M_{K}^*)_n$, $b \in g_{-}. This means that the operators $\pi_{M_{K}}^*(g)$ are continuous for any $g \in g_{+}$ and
$\pi_M(g) \subset \text{End}M^*_g$ is an equicontinuous family of operators. It remains to show that the assignment $g \mapsto \pi_M(g)$ is continuous for $g \in g$. Since $g_*$ is discrete, it is enough to check this statement for $g \in g_-$.

Let $f \in M_*$. Let $f_0$ be the reduction of $f$ modulo $(M^*_g)_H$. We can regard $f$ as an element of $\bigoplus_{a \in a^*} \text{End}g_a$. Let us write $f_0$ in terms of the basis $\{a_i\}$, and let $T_\alpha(f)$ be the set of all $\alpha \in H$ such that $a_i$ is involved in this expression.

Let $S_\alpha(g)$ be as in the proof of Lemma 7.3, and $i \in I \setminus S_\alpha(T_{a_i}(f))$. Then it is easy to see that $bf \in (M^*_g)_H$. This shows that for any $n \geq 0$ and any $f \in M^*_g$, $bf \in (M^*_g)_H$ for almost all $i \in I$.

Thus, $M_*$ is an equicontinuous $g$-module. □

**Remark.** If $g_*$ is infinite dimensional then $M_*$ is not, in general, an equicontinuous $g$-module, since the family of operators $\{\pi_M(g); g \in g_*\}$ may fail to be equicontinuous.

7.6. The Casimir element.

Consider the tensor product $a \otimes a^*$. This space can be embedded into $\text{End}a$, by $(x \otimes f)(y) = f(y)x, x, y \in a, f \in a^*$. This embedding defines a topology on $a \otimes a^*$, obtained by restriction of the weak topology on $\text{End}a$. Let $a \otimes a^*$ be the completion of $a \otimes a^*$ in this topology. Since the image of $a \otimes a^*$ is dense in $\text{End}a$, this completion is identified with $\text{End}a$.

**Lemma 7.5.** Let $V, W \in M_0$. The map $\pi_V \otimes \pi_W : a \otimes a^* \to \text{End}(V \otimes W)$ extends to a continuous map $\alpha \otimes \alpha^* \to \text{End}(V \otimes W)$.

**Proof.** Let $x \in V \otimes W$ be a vector. It is easy to see that the map $\pi_V \otimes \pi_W(x) : a \otimes a^* \to V \otimes W$ is continuous. Since the space $V \otimes W$ is complete, this map extends to a continuous map $\alpha \otimes \alpha^* \to V \otimes W$. This allows us to define a linear map $\pi_V \otimes \pi_W : a \otimes a^* \to \text{End}(V \otimes W)$. We would like to show that this map is continuous;

Let $f \in V \otimes W$ be a vector, and $n \geq 0$ be an integer. Let $P \subset V \otimes W$ be an open subspace, and $U = \{A \in \text{End}(V \otimes W) : A \in P\}$. Since open sets of this form generate the topology on $\text{End}(V \otimes W)$, it is enough to show that there exists a neighborhood of zero $Y \subset a \otimes a^*$ such that $(\pi_V \otimes \pi_W)(Y) \subset U, i.e. (\pi_V \otimes \pi_W)(Y)x \subset P$.

We can assume that $P = V_1 \otimes W_1 \otimes V_2$, where $V_1, W_1$ are open subspaces of $V, W$. By the equicontinuity of $\pi_V(g), \pi_W(g), g \in g$, there exist open subspaces $V_2 \subset V, W_2 \subset W$ such that $\pi_V(g)V_2 \subset V_1, \pi_W(g)W_2 \subset W_1$. Let $y \in V \otimes W$ be a vector in the usual tensor product of $V$ and $W$ such that $y - x \in V_2 \otimes W_2 \otimes V_2$. Then for any $f \in a \otimes a^*$, $(\pi_V \otimes \pi_W)(Y)(y - x) \in P$, so it is enough to find $Y$ satisfying the condition $(\pi_V \otimes \pi_W)(Y)y \subset P$.

We have $y = \sum_{j=1}^m v_j \otimes w_j$, $v_j \in V, w_j \in W$. Let $X \subset a$ be a finite-dimensional subspace such that for any $b \in X \subset a^*$ but $b_j \in W$, for $j = 1, \ldots, m$. Such a subspace exists by Lemma 7.1. The set $Y = a \otimes X$ (the completion of $a \otimes X$ in $a \otimes a^*$) is open in $a \otimes a^*$, and $(\pi_V \otimes \pi_W)(Y)y \subset P$, as desired. This shows the continuity of $\pi_V \otimes \pi_W$ on $a \otimes a^*$, □

Let $r \in a \otimes a^*$ be the vector corresponding to the identity operator under the identifications $a \otimes a^*$ with $Enda$. Let $r^{op} \in a^* \otimes a$ be the element obtained from $r$ by permutation of the components. We define the Casimir element $\Omega \in a \otimes a^* \otimes a^* \otimes a$ to be the sum $r \otimes r^{op}$.

It is easy to see that $r = \sum a_i \otimes b_i, r^{op} = \sum b_i \otimes a_i, \Omega = \sum (a_i \otimes b_i \otimes a_i \otimes b_i)$. The proof is complete.
Let $V, W$ be equicontinuous $g$-modules, and denote by $\pi_V : g \to \text{End} V$, $\pi_W : g \to \text{End} W$ the corresponding linear maps. Let $\Omega_{VW} = \pi_V \otimes \pi_W(\Omega)$. This endomorphism of $V \otimes W$ is well defined and continuous by Lemma 7.5. Moreover, it is easy to see that $\Omega_{VW}$ commutes with $g$, so it is an endomorphism of $V \otimes W$ as an equicontinuous $g$-module.

**Remark.** Although the Casimir operator $\Omega = \sum (a_i \otimes b_i + b_i \otimes a_i)$ is defined in the product of any two equicontinuous $g$-modules $V \otimes W$, the Casimir element $C = \sum (a_i b_i + b_i a_i)$ in general (for $\dim a = \infty$) has no meaning as an operator in an equicontinuous $g$-module $V$.

7.7. Dribnfeld category. Let $\mathcal{M}^a$ denote the category whose objects are equicontinuous $g$-modules, and $\text{Hom}_{\mathcal{M}^a}(U, W) = \text{Hom}_g(U, W)[[A]]$. This is an additive category. For brevity we will later write $\text{Hom}$ for $\text{Hom}_{\mathcal{M}^a}$.

Define a structure of a braided monoidal category on $\mathcal{M}^a$ analogously to Section 1.4, using an associator $\Phi$ and the functor $\otimes$. As before, we identify $\mathcal{M}^a$ with a strict category and forget about positions of brackets.

Let $\gamma$ be the functorial isomorphism defined by $\gamma_{XY} = \delta_{X,Y} \in \text{Hom}_{\mathcal{M}^a}(X \otimes Y, Y \otimes X)$, $X, Y \in \mathcal{M}^a$. It is easy to check that $\gamma$ is a braiding on $\mathcal{M}^a$. We will need the braiding $\gamma$ in our construction below.

8. The fiber functor.

8.1. The category of complete $k[[h]]$-modules.

Let $V$ be a complete vector space over $k$. Then the space $V[[h]] = \bigoplus V \cdot [h]^n$ of formal power series in $h$ with coefficients in $V$ is also a complete vector space. Moreover, $V[[h]]$ has a natural structure of a topological $k[[h]]$-module. We call a topological $k[[h]]$-module complete if it is isomorphic to $V[[h]]$ for some complete $V$.

Let $\mathcal{A}^c$ be the category of complete $k[[h]]$-modules, where morphisms are continuous $k[[h]]$-linear maps. It is an additive category. Define the tensor structure on $\mathcal{A}^c$ as follows. For $V, W \in \mathcal{A}^c$ define $V \otimes W$ to be the quotient of the completed tensor product $V \otimes W$ by the image of the operator $h \otimes 1 - 1 \otimes h$. It is clear that for $V, W \in \mathcal{A}^c$, $V \otimes W$ is also in $\mathcal{A}^c$. The category $\mathcal{A}^c$ equipped with the functor $\otimes$ is a symmetric monoidal category.

Let $\text{CVect}$ be the category of complete vector spaces. We have the functor of extension of scalars, $V \mapsto V[[h]]$, acting from $\text{CVect}$ to $\mathcal{A}^c$. This functor respects the tensor product, i.e. $(V \otimes W)[[h]]$ is naturally isomorphic to $V[[h]] \otimes W[[h]]$.


Let $(g, \rho, \rho_0)$ be a Manin triple, and $\mathcal{M}$ be the Dribnfeld category associated to $g$. Let $M_\alpha$, $\mathcal{M}_\alpha$ be the Verma modules over $g$ defined in Section 7.5.

Recall that the modules $M_\alpha$ are identified with $U(g)$. Thus, we can define the maps $i_\alpha : M_\alpha \to M_\alpha \otimes M_\alpha$ given by comultiplication in the universal enveloping algebra $U(g)$. These maps are $U(g)$-intertwiners, since they are $U(g)$-intertwiners and map the vector $1_k$ to the $g_2$-invariant vector $1_k \otimes 1_k$.

Let $\mathcal{M}_\alpha$ be as in Section 7.5, and $f, g \in \mathcal{M}_\alpha$. Consider the linear functional $M_\alpha \to k$ defined by $v \mapsto (f \otimes g)(I_+(v))$. It is easy to check that this functional is continuous, so it belongs to $M_\alpha^*$. Define the map $i_\alpha^* : M_\alpha^* \otimes M_\alpha^* \to M_\alpha$ by $i_\alpha^*(f \otimes g)(v) := (f \otimes g)(I_+(v))$, $v \in M_\alpha$. It is clear that $i_\alpha^*$ is continuous, so it extends to a morphism in $\mathcal{M}^a$: $i_\alpha^* : M_\alpha^* \otimes M_\alpha^* \to M_\alpha$.

Let $V \in \mathcal{A}^c$ Consider the space $\text{Hom}_g(M_\alpha, M_\alpha^*)$, where $\text{Hom}_g$ denotes the
set of continuous homomorphisms. Equip this space with the weak topology (see Section 7.1).

Lemma 8.1. The complete vector space $\text{Hom}_q(M_\ast, M_\ast \otimes V)$ is isomorphic to $V$. The isomorphism is given by $f \mapsto (1_+ \otimes 1)(f(1_+))$, $f \in \text{Hom}_q(M_\ast, M_\ast \otimes V)$.

Proof. By the Frobenius reciprocity $\text{Hom}_q(M_\ast, M_\ast \otimes V)$ is isomorphic, as a topological vector space, to the space of invariants $(M_\ast \otimes V)^{\ast}$, via $f \mapsto f(1_-)$. Consider the space $\text{Hom}_q(M_\ast, V)$ of continuous homomorphisms from $M_\ast$ to $V$, equipped with the weak topology, and the map $\phi : (M_\ast \otimes V) \rightarrow \text{Hom}_q(M_\ast, V)$, given by $\phi(f \otimes v)(\omega) = f(v)\omega, v \in M_\ast, \omega \in V$. It is clear that $\phi$ is injective and continuous.

Claim. The map $\phi$ restricts to an isomorphism $(M_\ast \otimes V)^{\ast} \rightarrow \text{Hom}_q(M_\ast, V)$.

Proof. It is clear that $\phi((M_\ast \otimes V)^{\ast}) \subset \text{Hom}_q(M_\ast, V)$. So it is enough to show that any continuous $g_\ast$-intertwiner $g : M_\ast \rightarrow V$ is of the form $\phi(g')$, $g' \in (M_\ast \otimes V)^{\ast}$, where $g'$ continuously depends on $g$.

Let $X \subset V$ be an open subspace. Then for any $g_\ast$-intertwiner $g : M_\ast \rightarrow V$ and $n \geq 1$ the image of $g(U_n(\varphi_n)1_\ast)$ in $V/V_m$ is finite-dimensional. This shows that $g = \phi(g')$ for some $g' \in (V \otimes M_\ast)^{\ast}$. It is clear that $g'$ is continuous in $g$. The claim is proved.

By the Frobenius reciprocity, the space $\text{Hom}_q(M_\ast, V)$ is isomorphic to $V$ as a topological vector space, via $f \mapsto f(1_-)$. The lemma is proved. □

8.3. The forgetful functor.

Let $F : \mathcal{M} \rightarrow \mathcal{A'}$ be a functor given by $F(V) = \text{Hom}(M_\ast, M_\ast \otimes V)$. Lemma 8.1 implies that this functor is naturally isomorphic to the “forgetful” functor which associates to every equicontinuous $g_\ast$-module $M$ the complete $k[[h]]$-module $M[[h]]$.

The isomorphism between these two functors is given by $f \mapsto (1_+ \otimes 1)(f(1_-))$, for any $f \in F(M)$. Denote this isomorphism by $\tau$.

8.4. Tensor structure on the functor $F$.

From now on, when no confusion is possible, we will denote the tensor product in the categories $\mathcal{M}$ and $\mathcal{A'}$ by $\otimes$, instead of $\otimes$ and $\otimes$.

Define a tensor structure on the functor $F$ constructed in Section 8.3.

For any $v \in F(V), w \in F(W)$ define $J_{v,W}(v \otimes w)$ to be the composition of morphisms:

$$
M_\ast \xrightarrow{1_+} M_\ast \otimes M_\ast \xrightarrow{v \otimes w} M_\ast \otimes V \otimes M_\ast \otimes W \xrightarrow{1_+ \otimes 1_+ \otimes 1_+} (V \otimes M_\ast \otimes W)
$$

where $1_+ \otimes 1_+ \otimes 1_+$ denotes the braiding $1_+$ acting in the second and third components of the tensor product. That is,

$$
J_{v,W}(v \otimes w) = (1_+ \otimes 1_+ \otimes 1_+) \circ (1_+ \otimes 1_+ \otimes 1_+) \circ (v \otimes w) \circ 1_+.
$$

Proposition 8.2. The maps $J_{v,W}$ are isomorphisms and define a tensor structure on the functor $F$.

Proof. It is obvious that $J_{v,W}$ is an isomorphism since it is an isomorphism modulo $h$. 
To prove the associativity of $J_{14}$, we need the following result.

**Lemma 8.3** \((i_+ \otimes 1) \circ i_- = (1 \otimes i_+) \circ i_- \in \text{Hom}(M_u, M_{13}^{[3]}); (i^*_+ \otimes 1) \circ i^*_+ = (1 \otimes i^*_+) \circ i^*_+ \in \text{Hom}(M_{24}, (M_1^{[2]})^{[3]})\).

**Proof.** The proof of the first identity coincides with the proof of Lemma 2.3 in Part I. To prove the second identity, define $M_u \otimes M_3 \otimes M_3$ to be space of continuous linear functionals on $M_3 \otimes M_3 \otimes M_3$. Since the operators $\Omega_{i_+} \in \text{End}_{M_3}(M_3 \otimes M_3 \otimes M_3)$ are continuous, one can define the dual operators $\Omega^{*_{i_+}} \in \text{End}_{M_3}(M_3 \otimes M_3 \otimes M_3)$, and hence the operator $\Phi^*$ dual to $\Phi$. It is easy to show analogously to the proof of Lemma 3.3 that $\Phi^*(1_+ \otimes 1_+ \otimes 1_+) = 1_+ \otimes 1_+ \otimes 1_+$, which implies the second identity of Lemma 8.3.

Now we can finish the proof of the proposition. We need to show that for any $v \in F(V)$, $w \in F(W)$, $u \in F(U)$ $J_{14} \circ (J_{12} \circ (e \otimes w \otimes u) = J_{14} \circ (1 \otimes J_{12}) \circ (v \otimes w \otimes u)$, i.e.,

\[
(\gamma_{14} \otimes 1 \otimes 1) \circ (\gamma_{12} \otimes 1 \otimes 1) \circ (i_- \circ (e \otimes w \otimes u) \circ (i_- \otimes 1) \circ i_- = (8.3)
\]

in $F(V \otimes W \otimes U)$, where $\gamma_{12}$ means the braiding applied to the product of the first and the second factors and to the fourth factor. Because of Lemma 8.3 and covariance relation of $\gamma_{12}$ and $i^*_+ \otimes 1 \otimes 1 \otimes 1$, identity (8.3) is equivalent to the identity

\[
(\gamma_{14} \otimes 1 \otimes 1) \circ (\gamma_{12} \otimes 1 \otimes 1) \circ (i_- \circ (e \otimes w \otimes u) \circ (i_- \otimes 1) \circ i_- = (8.4)
\]

in $\text{Hom}(M_{24} \otimes V \otimes M_3 \otimes W \otimes M_3 \otimes U, M_1 \otimes V \otimes W \otimes U)$.

To prove this equality, we observe that the functoriality of the braiding implies the identity

\[
\gamma_{12} \circ (1 \otimes i^*_+ \otimes 1 \otimes 1) = (1 \otimes i^*_+ \otimes 1 \otimes 1) \circ \gamma_{12}
\]

Using $(8.5)$ and the identity $(i^*_+ \otimes 1) \circ i^*_+ = (1 \otimes i^*_+ \circ i^*_+$, which follows from Lemma 8.3, we reduce (8.4) to the identity $\gamma_{14} \otimes 1 \otimes 1 = \gamma_{12}$, which follows directly from the coassociativity axioms. $\square$

We will call the functor $F$ equipped with the tensor structure defined above the fiber functor.

**9. Quantization of Lie bialgebras.**

9.1. The algebra of endomorphisms of the fiber functor.

Let $H = \text{End}(F)$ be the algebra of endomorphisms of the fiber functor $F$, with a topology defined by the ideal $\mathfrak{h} \subset H$. It is clear that $H$ is a topological algebra over $k[[h]]$ (see Part I, Section 3.1).

Let $H_0$ be the algebra of endomorphisms of the forgetful functor $M_0^* \to CV_{vet}$.

It follows from Lemma 8.1 that the algebra $H$ is naturally isomorphic to $H_0[[h]]$.

Let $F^* = M^* \times M^* \to A^*$ be the bifunctor defined by $F^*(V, W) = F(V) \otimes F(W)$.

Let $H^* = \text{End}(F^*)$. It is clear that $H^* \cong H \otimes H$ but $H^* \neq H \otimes H$ unless $g$ is finite dimensional.
The algebra $H$ has a natural “comultiplication” $\Delta : H \rightarrow H^2$ defined by $\Delta(a)_{\alpha \beta} = \sum_{a^1 \in A, a^2 \in A} \nu_{\alpha \beta}(a^{12})$, where $a^1 \in H \otimes H \otimes H$ and $a^2 \in H$ are the elements of $H$ and $H^2$, respectively. We can also define the counit $\varepsilon$ on $H$ by $\varepsilon(a) = a_1 \in k[[h]]$, where 1 is the neutral object.

A topological algebra $A$ over $k[[h]]$ is said to be a topological bialgebra if it is equipped with a coassociative coproduct $\Delta : A \rightarrow A \otimes A$ (where $\otimes$ is the tensor product in $A$) and a counit $\varepsilon : A \rightarrow k[[h]]$ which are continuous, and satisfy the standard axioms of a bialgebra.

We will need the following statement.

**Proposition 9.1.** Let $A \subseteq H$ be a topological subalgebra such that $\Delta(A) \subseteq A \otimes A$. Then $(A, \Delta, \varepsilon)$ is a topological bialgebra over $k[[h]]$.

The proof is straightforward.

**Remark.** For infinite-dimensional $g$, the algebra $H$ equipped with the topology defined by the ideal $hH$ is not a topological bialgebra since $\Delta(h) = h \otimes h$ is not a subset of $H \otimes H$.

In the following sections, we construct a quantum universal enveloping algebra $U_q(g)$, which is a quantization of the Lie bialgebra $g_+$ in the sense of Drinfeld (see [Dr] and Part I, Section 3.1). Namely, the algebra $U_q(g_+)$ is obtained as a subalgebra of $H$ such that $\Delta(A) \subseteq A \otimes A$.

**9.2. The algebra $U_q(g_+)$.**

Let $\tau \in F(M_+)$. Define the automorphism $m_\tau(x)$ of the functor $F$ as follows. For any $V \in M^+$, $v \in F(V)$, define the element $m_\tau(x)v \in F(V)$ by the composition of the following morphisms in $M$: $m_\tau(x)v = (i_+ \otimes 1) \circ (1 \otimes v) \circ x$. This defines a linear map $m_\tau : F(M_+) \rightarrow H$. Denote the image of this map by $U_q(g_+)$. It is easy to see that for any $a \in U_q(g_+)$, $m_\tau(a)_{\alpha \beta} v = a \tau(v) \bmod h$, which implies that $m_\tau$ is an embedding.

**Proposition 9.2.** $U_q(g_+)$ is a subalgebra in $H$.

**Proof.**

Using Lemma 8.3, for any $x, y \in F(M_+)$, $v \in M^+$, $x \in F(V)$ we obtain

$$m_\tau(x)m_\tau(y)v = (i_+ \otimes 1) \circ (1 \otimes i_+ \otimes 1) \circ (1 \otimes 1 \otimes v) \circ (1 \otimes y) \circ x =$$

$$= (i_+ \otimes 1) \circ (i_+ \otimes 1 \otimes 1) \circ (1 \otimes 1 \otimes v) \circ (1 \otimes y) \circ x =$$

$$= (i_+ \otimes 1) \circ (1 \otimes v) \circ (i_+ \otimes 1 \otimes 1) \circ (1 \otimes y) \circ x =$$

$$= (i_+ \otimes 1) \circ (1 \otimes v) \circ x,$$

where $x = (i_+ \otimes 1) \circ (y \otimes 1) \circ x \in F(M_+)$. So by the definition we get $m_\tau(x)m_\tau(y) = m_\tau(x)$. □

Note that the algebra $U_q(g_+)$ is a deformation of the algebra $U_q(g)$. Indeed, we can define a linear isomorphism $\mu : U_q(g_+)[[h]] \rightarrow U_q(g_+)$ by $\mu(a) = m_\tau(a_{1-})$. This isomorphism has the property $\mu(ab) = \mu(a) \circ \mu(b) \bmod h^2$, which follows from the fact that $\Phi = 1 \bmod h$, but in general $\mu(ab) \neq \mu(a) \circ \mu(b)$.

The subalgebra $U_q(g_+)$ has a unit which is equal to $\mu(1)$, $1 \in U_q(g_+)$. To check this, it is enough to observe that $\mu(1)$ is invertible and check the identity $\mu(1)^2 = \mu(1)$.

**9.3. The coproduct on $U_q(g_+)$.**
Proposition 9.3. The algebra $U_q(\mathfrak{g}_e)$ is closed under the coproduct $\Delta$, i.e. $\Delta(U_q(\mathfrak{g}_e)) \subset U_q(\mathfrak{g}_e) \otimes U_q(\mathfrak{g}_e)$, and for any $x \in F(M_e)$ one has

\begin{equation}
\Delta(m_+(x)) = (m_+ \otimes m_+)(J_{M_e}^{-1}((1 \otimes \iota_+) \circ x)).
\end{equation}

Proof. Let $x \in F(M_e), V, W \in \mathcal{M}_e, v \in V, w \in W$. By the definition of $\Delta$ and $m_+$, the element $\Delta(m_+(x)) \in H^2$ is uniquely determined by the identity

\begin{equation}
(c_{-i}^+ \otimes 1 \otimes 1) \circ (1 \otimes c_{-i}^+ \otimes 1 \otimes 1) \circ \gamma_{23} \circ (1 \otimes v \otimes w) \circ R(1 \otimes \iota_+) \circ x =
\end{equation}

in $F(V \otimes W)$.

The element $X = J_{M_e}^{-1}((1 \otimes \iota_+) \circ x) \in F(M_e) \otimes F(M_e)$ is, by the definition, uniquely determined by the identity

\begin{equation}
(1 \otimes \iota_+) \circ x \in (c_{-i}^+ \otimes 1 \otimes 1) \circ \gamma_{23} \circ X \circ \iota_-
\end{equation}

in $F(M_e \otimes M_e)$. Therefore, to prove formula (9.2), it is enough to prove the equality obtained by substitution of $((c_{-i}^+ \otimes 1 \otimes 1) \circ (1 \otimes v \otimes 1 \otimes w) \circ X)$ instead of $\Delta(m_+(x))(v \otimes w)$ in (9.3):

\begin{equation}
(c_{-i}^+ \otimes 1 \otimes 1) \circ (1 \otimes c_{-i}^+ \otimes 1 \otimes 1) \circ \gamma_{23} \circ (1 \otimes v \otimes w) \circ (1 \otimes \iota_+) \circ x =
\end{equation}

in $F(V \otimes W)$.

Using the functoriality of the braiding and Lemma 8.3, we obtain

\begin{equation}
(c_{-i}^+ \otimes 1 \otimes 1) \circ \gamma_{23} \circ ((c_{-i}^+ \otimes 1 \otimes 1) \circ (1 \otimes v \otimes 1 \otimes w) =
\end{equation}

in $\text{Hom}(M_e \otimes M_e, M_e \otimes M_e, M_e \otimes V \otimes W)$. It is easy to see that $\iota_+^* \circ \gamma = \iota_+^*$, so using Lemma 8.3 again, we get from (9.6):

\begin{equation}
(c_{-i}^+ \otimes 1 \otimes 1) \circ (1 \otimes c_{-i}^+ \otimes 1 \otimes 1) \circ \gamma_{23} \circ (1 \otimes v \otimes w) \circ (1 \otimes \iota_+) \circ x =
\end{equation}

Substituting (9.7) into the right hand side of (9.5) and using (9.4), we get

\begin{equation}
(c_{-i}^+ \otimes 1 \otimes 1) \circ \gamma_{23} \circ ((c_{-i}^+ \otimes 1 \otimes 1) \circ (1 \otimes v \otimes 1 \otimes w) \circ X \circ \iota_- =
\end{equation}

in $F(V \otimes W)$, which proves (9.2). The proposition is proved. \square
Corollary 9.4. The algebra $U_h(g_+)$, equipped with the coproduct $\Delta$, is a quantized universal enveloping algebra.

Proof. It follows from Lemma 9.1 and Propositions 9.2, 9.3 that $U_h(g_+)$ is a topological bialgebra over $k[[\hbar]]$ isomorphic to $U(g_-)[[\hbar]]$ as a topological $k[[\hbar]]$-module, and such that $U_h(g_+)$ is isomorphic to $U(g_-)$ as a bialgebra. This implies that $U_h(g)$ has an antipode, because the antipode exists mod $h$. Thus, $U_h(g_+)$ is a quantized universal enveloping algebra. □

9.4. The algebra $U_h(g_+)$ is a quantization of $g_+$.

Proposition 9.5. The algebra $U_h(g_+)$ is a quantization of the Lie bialgebra $g_+$.

Proof. Let $x \in U_h(g_+)$ be such that there exists $x_0 \in g_+ \subset U(g_+)$ satisfying the condition $x \equiv x_0 \mod h$.

It is easy to show that for any $V, W \in M$

$$\tau_{V, W} \circ J_{V, W} \circ (\tau_{W, V}^{-1}) = 1 + \hbar r/2 + O(\hbar^2)$$

in $\text{End}(V \otimes W)$. From (9.9) and the definition of coproduct, analogously to the proof of Proposition 3.6 in Part I, it is easy to obtain the congruence

$$h^{-1}(\Delta(x) - \Delta^0(x)) = \delta(x_0) \mod h.$$ 

which means that $U_h(g_+)$ is the quantization of $g_+$. □

Thus, we have proved the following theorem, which answers question 1.1 in [Dr3].

Theorem 9.6. Let $g$ be a Lie bialgebra over $k$. Then there exists a quantized universal enveloping algebra $U_h(g)$ over $k$ which is a quantization of $g$.

9.5. The isomorphism between two constructions of the quantization.

Let us compare the results of the previous sections to the results of Part I. In Part I, we showed the existence of quantization for any finite-dimensional Lie bialgebra. Let $(g, g_-, g_+)$ be a finite-dimensional Manin triple. Let $U_h(g_+)$ denote the quantization of $g_+$ constructed in this section, and by $U_h(g_-)$ the quantization constructed in Part I.

Proposition 9.7. The quantized universal enveloping algebras $U_h(g_+), U_h(g_-)$ are isomorphic.

Proof. If $g$ is finite-dimensional, then $M_+$ is an equicontinuous $g$-module. Let $\hat{F} : M^+ \to \hat{A}^+$ be the functor defined by $\hat{F}(V) = \text{Hom}(M_+ \otimes M_-, V), V \in M^+$.

The tensor structure on $\hat{F}$ can be defined as in Part I.

Let $\sigma \in \text{Hom}(1, M_+ \otimes M_-)$ be the canonical element. Consider the morphism $\chi : \hat{F} \to F$, defined as follows. For any $V \in M_-, v \in \hat{F}(V)$, define $\chi_V(v) \in F(V)$ as the composition $\chi_V(v) = (1 \otimes v) \circ (\sigma \otimes 1)$. It is obvious that $\chi$ is an isomorphism of additive functors.

Claim. $\chi$ is an isomorphism of tensor functors.

Proof. The statement is equivalent to the identity

$$\chi(V)(1 \otimes v) \circ \gamma_{V} \circ (1 \otimes i_+ \otimes l_v) \circ (\sigma \otimes 1) =$$

$$\gamma_{V} \circ (1 \otimes i_+ \otimes l_v) \circ (\sigma \otimes 1 \otimes \sigma \otimes 1) \circ i_+. (9.11)$$
which should be satisfied in $\text{Hom}(M_r \otimes M_s \otimes V \otimes W)$ for any $V, W \in M_r$, $v \in \tilde{F}(V)$, $w \in \tilde{F}(W)$. Using the identity $(1 \otimes v \otimes 1 \otimes w) \circ \gamma_{23} = \gamma_{23,4} \circ (1 \otimes 1 \otimes v \otimes w)$, we reduce (9.11) to the identity

\[
\beta_{34} \circ (1 \otimes i_r \otimes i_w) \circ (\sigma \otimes 1) = (\gamma_{12,3} \otimes 1 \otimes 1 \otimes 1) \circ \gamma_{23,4} \circ (1 \otimes 1 \otimes \sigma \otimes 1) \circ i_w.
\]

(9.12)

in $\text{Hom}(M_r \otimes M_s \otimes M_t \otimes M_u \otimes M_r \otimes M_u)$. Moving $\beta_{34}$ from left to right and interchanging $\beta_{34}^{1}$ with $i_r \otimes 1 \otimes 1 \otimes 1$, so that (9.12) is equivalent to the identity:

\[
(1 \otimes i_r \otimes i_w) \circ (\sigma \otimes 1) = (\gamma_{12,3} \otimes 1 \otimes 1 \otimes 1) \circ \gamma_{23,4} \circ (1 \otimes 1 \otimes \sigma \otimes 1) \circ i_w.
\]

(9.13)

in $\text{Hom}(M_r \otimes M_s \otimes M_t \otimes M_u \otimes M_r \otimes M_u)$. It is clear that $\gamma_{12,3} \circ (1 \otimes \sigma) = \sigma \otimes 1$ in $\text{Hom}(M_r \otimes M_s \otimes M_t \otimes M_u)$. Therefore, using the relations $\gamma_{12,3} \gamma_{23,4} = \gamma_{23,4} \gamma_{12,3}$, and $\beta_{23} = 1$, we reduce (9.13) to

\[
(1 \otimes i_r) \circ \sigma = (i_r \otimes 1 \otimes 1) \circ \gamma_{23} \circ (\sigma \otimes \sigma)
\]

(9.14)

in $\text{Hom}(1, M_s \otimes M_t \otimes M_u \otimes M_r)$. Since $i_r \circ \gamma = i_r \gamma$, we can rewrite (9.14) as

\[
(1 \otimes i_r) \circ \sigma = (i_r \otimes 1 \otimes 1) \circ \gamma_{12,3} \circ (\sigma \otimes \sigma).
\]

(9.15)

Using the equality $\gamma_{12,3} \circ (\sigma \otimes 1) = 1 \otimes \sigma$, we reduce (9.15) to

\[
(1 \otimes i_r) \circ \sigma = (i_r \otimes 1 \otimes 1) \circ (1 \otimes \sigma \otimes 1) \circ \sigma.
\]

(9.16)

To prove this equality, we compute the image of $1 \otimes 1$ under right hand side of (9.16). In this calculation, we can ignore the action of the associator because for any representations $V_1, V_2, V_3$ of $g$ the associator acts trivially on the $g$-invariants in $V_1 \otimes V_2 \otimes V_3$. The calculation yields that $1$ goes to $(1 \otimes i_r) \circ (\sigma(1))$, which proves (9.16). The claim is proved.

Let $M \subset M^\sigma$ be the full subcategory of discrete $g$-modules, and $\tilde{U}_h(g) = \text{End}(\tilde{F}(M))$ be the quantization of $g$ constructed in Part I. It is easy to show that the homomorphism of topological Hopf algebras $\tilde{U}_h(g) \to \tilde{U}_h(g)$ defined by restriction from $M^\sigma$ to $M$ is an isomorphism. Since both algebras are canonically isomorphic to $U(g)[[\hbar]]$. This means that the morphism $\chi$ defined above induces an isomorphism of topological Hopf algebras $\tilde{U}_h(g) \to U_h(g)$. It is easy to check that this isomorphism maps $\tilde{U}_h(g_+)$ onto $U_h(g_+)$, which proves the proposition. ~\(\square\)

10. Universality and functoriality of the quantization of Lie bialgebras.

10.1. Ayclic functions.

Let $V$ be vector space over $k$. For any integers $m, n \geq 0$, let $H_{cm} = \text{Hom}(V^m, V^m)$ be the space of tensors of rank $m, n$ on $V$.

Let $B = \oplus_{m,n \geq 0} H_{cm}$. We have two binary operations on $B$: the tensor product and the composition. (If the composition makes no sense, we set it to zero.)
Let $m_1, \ldots, m_r, n_1, \ldots, n_r$ be nonnegative integers, and $W = \otimes_{i=1}^{r} H_{m_i n_i}$. Let $p_i : W \to H_{m_i n_i}$ be the natural projections.

Let $X$ be a subset of $W$ and $Y_X$ be the space of all functions from $X$ to $B$. Denote by $A_X$ the smallest subspace in $Y_X$ closed under composition and tensor product, and satisfying the following conditions:

(i) $p_i | X \in A_X$, $i = 1, \ldots, r$

(ii) If $\sigma \in Y_X$ is a permutation operator from $S_p \subset H_{pp}$ regarded as a constant function on $X$, then $\sigma \in A_X$.

We call an element of $A_X$ an acyclic function on $X$.

10.2. Universal quantization.

Let $g$ be a Lie bialgebra over $k$, and $U_h(g)$ be its quantization constructed in Chapter 9. Recall that it is a $k[[h]]$-module, $U_h(g)$ was identified with $U(g)[[h]]$, which, in turn, we identify with $S_h[g]$ in the standard way. Therefore, the multiplication map $\rho : U_h(g) \otimes U_h(g) \to U_h(g)$ splits in a direct sum of linear maps $\rho_{\mu_{i,\delta}} : S^{\mu_{i,\delta}} g \otimes S^{\nu_{i,\delta}} g \to S^{\nu_{i,\delta}} g[[h]]$. Similarly, the coproduct $\Delta : U_h(g) \to U_h(g) \otimes U_h(g)$ splits in a direct sum of linear maps $\Delta_{\mu_{i,\delta}} : S^{\mu_{i,\delta}} g \to S^{\mu_{i,\delta}} g \otimes S^{\mu_{i,\delta}} g[[h]]$. All these linear maps are functions of the commutator $[,]$ and cocommutator $\delta$ of the Lie bialgebra $g$.

Now consider the setting of Section 10.1, with $n_1 = 2, m_2 = 1, n_2 = 1, n_2 = 2$, $W = \text{Hom}(V \otimes V, V) \otimes \text{Hom}(V, V \otimes V), X \subset W$ the set of all pairs $(\mu, \delta) \in W$ satisfying the axioms of a Lie bialgebra $g$. To every Lie bialgebra $g \in X (g = (V, \mu, \delta))$, we have associated a quantized universal enveloping algebra $U_h(g)$, which is identified with $S_h[V][[h]]$ as a $k[[h]]$-module. Thus, we can regard $\rho_{\mu_{i,\delta}}, \Delta_{\mu_{i,\delta}}$ as functions on $X$ with values in $H_{m_{i+1}n_i}, H_{m_{i+1}n_i}[[h]]$, respectively.

**Theorem 10.1.** The coefficients of the $h$-expansion of $\rho_{\mu_{i,\delta}}, \Delta_{\mu_{i,\delta}}$, are acyclic functions on $X$.

**Remark.** Drinfeld calls a quantization having this property a universal quantization. Thus, the quantization of Lie bialgebras constructed in Chapter 9 is universal.

Theorem 10.1 implies functoriality of quantization. Namely, let $\text{LBA}$ denote the category of Lie bialgebras over $k$, and $\text{QUEA}$ denote the category of quantum universal enveloping algebras over $k[[h]]$.

**Theorem 10.2.** There exists a functor $Q : \text{LBA} \to \text{QUEA}$ such that for any $g \in \text{LBA}$ we have $Q(g) = U_h(g)$.

**Proof.** On objects, the functor $Q$ is already defined. Now let us define it on morphisms. Let $f : g_1 \to g_2$ be a homomorphism of Lie bialgebras. It defines a linear map $Q(f) : S_{g_1}[[h]] \to S_{g_2}[[h]]$. By Theorem 10.1, this map defines a homomorphism of Hopf algebras $U_h(g_2) \to U_h(g_1)$. The theorem is proved. □

10.3. Proof of Theorem 10.1. Let $g_0$ be a Lie bialgebra, $U_h(g_0)$ be its quantization. We will use the notation of Section 9.

By the definition, $U_h(g_0) = F(M_\omega) = \text{Hom}(M_\omega, M_\omega \hat{\otimes} M_\omega)$. We have the identifications $\xi : S_{g_0} \to M_\omega$ given by

$$\xi_g(\text{Sym}(x_1 \otimes \cdots \otimes x_r)) = \text{Sym}(x_1, \ldots, x_r) \xi$, $x_i \in g_0$.\]
and $\theta : U_k(\mathfrak{g}_+^r) \to M_\omega[[h]]$ by

$$\theta(x_i) = (1_i \otimes 1)(x_i 1_-).$$

They give us identifications $\eta = \theta^{-1} \xi_+ : S_{\mathfrak{g}_+^r}[h] \to U_k(\mathfrak{g}_+^r)$, and $\xi_+ : M_{\omega}^r \to S_{\mathfrak{g}_+^r}$ (here half denotes the completion by degree). From now on we fix these identifications and thus regard the spaces $M_{\omega}^r, U_k(\mathfrak{g}_+^r)$ as sums of spaces of the form $S_{\mathfrak{g}_+^r}[h]$ or $S_{\mathfrak{g}_+^r}[h]$. This allows us to make the statement that certain maps between tensor products of these spaces, depending on $[\cdot], \beta$, are acyclic functions (on $X$).

To prove the theorem, we need to show that the maps $\mu, \Delta, S$ are acyclic.

To implement the proof, we need a few Lemmas.

Let $r \in \mathfrak{g}_+^r \oplus \mathfrak{g}_-^r$ be the classical $r$-matrix, so that $\Omega = r + r^{op}$.

**Lemma 10.3.** (i) The map $\tau : M_\omega^r \otimes M_\omega 	o M_\omega \otimes M_\omega$ is acyclic.

(ii) The maps $r, r^{op} : M_\omega^r \otimes M_\omega \to M_\omega^r \otimes M_\omega$ are acyclic.

**Proof.** (i) For any nonnegative integers $m, n$ consider the mapping $\mathfrak{g}_+^r \otimes \mathfrak{g}_-^r \to S_{\mathfrak{g}_+^r} \otimes S_{\mathfrak{g}_-^r}$ given by

$$x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n \mapsto r(x_1 \ldots x_m 1_- \otimes y_1 \ldots y_n 1_-).$$

We need to show that this mapping is an acyclic function. We can do this by induction in $N = m + n$. If $N = 0$, the operator is zero and the statement is clear.

Assume the statement is proved for $N = K - 1$ and let us prove it for $N = K$.

Using the relation $[x \otimes 1 + 1 \otimes x, r] = \delta(x)$, $x \in \mathfrak{g}_+^r$, we can reduce the question to the case $m = K, n = 0$. In this case, the map is again zero, Q.E.D.

(ii) By the same reasoning as in (i), we get the statement for $r$. For $r^{op}$, we reduce the question to proving that the map $M_\omega^r \to M_\omega^r \otimes M_\omega$ given by $v \mapsto r^{op}(v \otimes 1_-)$ is acyclic.

Let $u = \text{Sym}(u_1 \ldots u_n) 1_+ \in M_\omega^r, y_1, \ldots, y_m \in \mathfrak{g}_-^r$. Let us compute the expression

$$X = (u \otimes 1)(r^{op}(v \otimes 1_-)) \in M_\omega.$$ 

We get

$$X = -(r^{op}(u \otimes 1))(v \otimes 1_-) - \sum_{i} L(b^i, y_1, \ldots, y_m) 1_+ x_i 1_-$$

where $a_i, b^i$ are dual bases of $\mathfrak{g}_+, \mathfrak{g}_-$, and $L$ is a polynomial of commutators of $b^i, y_1, \ldots, y_m$ over $\mathbb{Q}$ which is symmetric in $b^i, y_1, \ldots, y_m$ and depends only on $m$.

Using the duality of $\mathfrak{g}_+$ and $\mathfrak{g}_-$, from (10.3) we get

$$X = \sum_{i} b^i \otimes y_1 \otimes \ldots \otimes y_m, D_{\Delta}(v) x_i 1_-$$

where $D_{\Delta}(v) \in S_{\mathfrak{g}_+^r}$ is a linear combination of iterated cocommutators applied to $v$. This implies that $r^{op}(v \otimes 1_-)$ is a linear combination of iterated cocommutators applied to $v$, so the map $r \mapsto r^{op}(v \otimes 1_-)$ is acyclic. Q.E.D.

For any $x \in M_\omega^r$, let $\psi_x : M_\omega^r \to M_\omega^r \otimes M_\omega$ be the morphism such that

$$(1_+ \otimes 1) \psi_x 1_- = x.$$
Lemma 10.4. The map \( M_\ast \otimes M_\ast \to M_\ast^2 \otimes M_\ast \) defined by \( x \otimes y \to \psi(x) \psi(y) \) is acyclic.

Proof. For any \( x \in M_\ast \), \( y \in U(g) \), we have \( \psi(x) \psi(y) = \Delta(y) \psi(x) \psi(y) \), where \( \Delta \) is the coproduct in \( U(g) \). Since the map \( \Delta \) is obviously acyclic, it suffices to show that the assignment \( x \to \psi(x) \) is an acyclic map \( M_\ast \to M_\ast^2 \otimes M_\ast \).

Let \( z \in U(g) \). Since the vector \( \psi(z) \) is \( g \)-invariant, we have

\[
(z_1 \otimes 1)(\psi(1)_{-1}) = (1_+ \otimes S_0(z))(\psi(1)_{-1}) = S_0(z)x,
\]

where \( S_0 \) is the antipode of \( U(g) \). Let \( z = \text{Sym}(z_1 \otimes \ldots \otimes z_n) \), \( x = \text{Sym}(x_1 \otimes \ldots \otimes x_n) \), \( x_j \in g \), \( x_i \in g_+ \). Computing the product \( S_0(z)x \), we see that it is the linear combination of products of expressions of the form \( x_i \) and \( Z = \left[ ad^* x_i \ldots ad^* x_j, z_1 \ldots z_k \right] \), applied to \( 1_+ \), where \( \left[ a \right] \) denotes the \( g_- \)-component of \( a \). Using the identity \( [z_0]_{+1} = (1 \otimes z, \delta(a)) \), we can rewrite \( Z \) in the form \( Z = \left[ z_1, (1 \otimes z_1)(ad x_1 \ldots ad x_j, \psi(1)_{-1})\right] \). This shows that the product \( S_0(z)x \), regarded as an element of \( S_0 g_+ \), can be represented as a linear combination of summands of the form

\[
(z_1 \otimes \ldots \otimes z_n \otimes 1^m, x'),
\]

where \( x' \in S_0 g_+ \) is a polynomial symmetric function of \( x_1, \ldots, x_n \), such that the assignment \( \text{Sym}(x_1 \otimes \ldots \otimes x_n) \to x' \) is acyclic. This proves the acyclicity of \( \psi(1)_{-1} \).

Now we can show the acyclicity of the product in \( U_\ast(g) \). According to Chapter 8, for any \( x, y \in M_\ast \),

\[
x \otimes y = (1_+ \otimes 1_+ \otimes 1)(\Phi^{-1}(1 \otimes \psi(y))(\psi(z))^{-1})
\]

Since \( \Phi^{-1} \) is a noncommutative formal series of \( M_{12}, M_{23} \), the acyclicity of the map \( x \otimes y \to x \otimes y \) follows from Lemmas 10.3 and 10.4.

To prove the acyclicity of the coproduct \( \Delta \), consider the linear operator \( J \in \text{End}(M_\ast \otimes M_\ast)[[b]] \) defined by

\[
J(x \otimes y) = (1_+ \otimes 1_+ \otimes 1)(\Phi^{-1}(1 \otimes \psi(y))(\psi(z))^{-1})
\]

According to Proposition 9.3, the coproduct on \( U_\ast(g) \), (when \( U_\ast(g) \otimes M_\ast \) is identified with \( M_\ast \)), is written in the form

\[
\Delta(x) = J^{-1}_- \cdot (x).
\]

The map \( J \) is acyclic by Lemmas 10.3 and 10.4. Therefore, \( \Delta \) is acyclic.

10.4. Universal quantization of quasitriangular Lie bialgebras.

Let \( g \) be a quasitriangular Lie bialgebra over \( k \), and \( U^q(g) \) be its quasitriangular quantization constructed in Chapter 6. As a \( k[[b]] \)-module, \( U^q(g) \) was identified with \( U(g)[[b]] \), which we identify with \( S(g)[[b]] \) in the standard way. Therefore, the multiplication map \( \mu : U^q(g) \otimes U^q(g) \to U^q(g) \) splits in a direct sum of linear maps \( \mu^q_{\ast} : S^r g \otimes S^r g \to S^r g[[b]] \). Similarly, the coproduct \( \Delta : U^q(g) \to U^q(g) \otimes U^q(g) \) splits in a direct sum of linear maps \( \Delta^q_{\ast} : S^r g \to S^r g \otimes S^r g[[b]] \), and the quantum \( R \)-matrix \( R \) splits in a direct sum of \( R_{\ast} : S^r g \otimes S^r g[[b]] \). All these linear maps are functions of the commutator \( \{ \cdot \} \) and the classical \( r \)-matrix \( r \) of \( g \).

37
Now consider the setting of Section 10.1, with \( m_1 = 2, m_2 = 0, n_1 = 1, n_2 = 2 \), \( W = \text{Hom}(V \otimes V, V) \oplus V \otimes V \) \( X \subset W \) the set of all pairs \((i, r) \in W\) satisfying the axioms of a quasitriangular Lie bialgebra. To every quasitriangular Lie bialgebra \( g \in X \) \((g = (V, \{., .\}, r)\), we have associated a quantized universal enveloping algebra \( U^q(g) \), which is identified with \( SV[[h]] \) as a \( k[[h]] \)-module. Thus, we can regard \( p^m_{r,m} \), \( \Delta^m_{p,m} \), \( R_{m,n} \) as functions on \( X \) with values in \( H^{m+n}[[h]] \), \( H^{m+n}[[h]] \), \( H^{m+n}[[h]] \), respectively.

**Theorem 10.5.** The coefficients of the \( h \)-expansion of \( p^m_{r,m} \), \( \Delta^m_{p,m} \), \( R_{m,n} \) are acyclic functions on \( X \).

**Proof.** The proof is analogous to the proof of Theorem 10.1. \( \square \)

Theorem 10.5 implies functoriality of quasitriangular quantization. Namely, let \( QTLBA \) denote the category of quasitriangular Lie bialgebras over \( k \), and \( QTQUEA \) denote the category of quasitriangular quantum universal enveloping algebras.

**Theorem 10.6.** There exists a functor \( Q^q : QTLBA \to QTQUEA \) such that for any \( g \in QTLBA \) we have \( Q^q(g) = U^q(g) \).

**Proof.** The proof is analogous to the proof of Theorem 10.2. \( \square \)

10.5. Universal quantization of classical \( r \)-matrices.

Let \( A \) be an associative algebra over \( k \) with unit, and \( r \in A \otimes A \) be a solution of the classical Yang-Baxter equation. In Chapter 5, we assigned to \((A, r)\) a solution of the quantum Yang-Baxter equation: \( R(r) \in A \otimes A[[h]] \).

Consider the setting of Section 10.1, with \( m_1 = 2, m_2 = 0, n_1 = 1, n_2 = 2 \), \( W = \text{Hom}(V \otimes V, V) \oplus V \otimes V, X \subset W \) the set of all triples \((i, r) \in W\) such that \( r \) is an associative product, 1 is a unit, and \( r \) satisfies the classical Yang-Baxter equation. To every \( A \in X \) \((A = (V, \{., .\}, r)\), we have associated a quantum \( R \)-matrix \( R(r) \in A \otimes A[[h]] \). Thus, we can regard \( R \) as a function on \( X \) with values in \( H^{2}[[h]] \).

**Theorem 10.7.** The coefficients of the \( h \)-expansion of \( R \) are acyclic functions on \( X \).

**Proof.** The theorem follows from Theorem 10.5. \( \square \)

Theorem 10.5 implies functoriality of quantization of classical \( r \)-matrices.

Namely, let us call a classical Yang-Baxter algebra \((A, r)\) a pair \((A, r)\), where \( A \) is an associative algebra with unit over \( k \), and \( r \in A \otimes A \) satisfies the classical Yang-Baxter equation, and a quantum Yang-Baxter algebra \((A, R)\) a pair \((A, R)\), where \( A \) is an associative algebra with unit over \( k \), and \( R \in A \otimes A[[h]] \) satisfies the quantum Yang-Baxter equation. Let \( CYBA, QYBA \) denote the categories of classical, respectively quantum, Yang-Baxter algebras. Morphisms in these categories are algebra homomorphisms preserving the unit and \( r \) (respectively, \( R \)).

**Theorem 10.8.** There exists a functor \( Q^q : CYBA \to QYBA \) such that for any \((A, r) \in CYBA \) we have \( Q^q(A, r) = (A, R(r)) \).

**Proof.** The proof is analogous to the proof of Theorem 10.2. \( \square \)

10.6. Quantization over a complete local \( q \)-algebra.
Let $K$ be a local Artinian or pro-Artinian commutative algebra over $\mathbb{Q}$, and $I$ be the maximal ideal in $K$. Let $k = K/I$ (it is a field of characteristic 0). Let $\mathbf{LBA}(K), \mathbf{QTLBA}(K), \mathbf{QUEA}(K), \mathbf{QTQUEA}(K)$ be the categories of Lie bialgebras, quasi-triangular Lie bialgebras, quantum universal enveloping algebras, quasi-triangular quantum universal enveloping algebras over $K$ which are topologically free as $K$-modules and cocommutative modulo $I$. Let $\mathbf{CYBA}(K), \mathbf{QYBA}(K)$ be the categories of classical Yang-Baxter algebras, quantum Yang-Baxter algebras, which are topologically free as $K$-modules and trivial modulo $I$ (i.e. $r = 0, R = 1$ modulo $I$).

In Section 10.2, we showed that $\mu, \delta$ are series of acyclic functions of $[\cdot, \cdot], h\delta$ with rational coefficients. Similarly, in Section 10.4, we showed that $\mu, \Delta, H$ are series of acyclic functions with rational coefficients of $[\cdot, \cdot], h\mu, h\delta$, and in Section 10.5 that $R$ is a series of acyclic functions with rational coefficients of $*, 1, h\delta$. Therefore, we can use these formalisms to define quantization over $K$ (the series will converge in the topology of $K$). This quantization over $K$ possesses the same functorial properties as the original quantization over $k[[h]]$. Thus, we obtain quantization functors $Q_A: \mathbf{LBA}(K) \rightarrow \mathbf{QUEA}(K), Q^+_A : \mathbf{QTLBA}(K) \rightarrow \mathbf{QTQUEA}(K), Q^+_K : \mathbf{CYBA}(K) \rightarrow \mathbf{QYBA}(K)$.

Appendix: computation of the product in $U_q(a)$ modulo $h^3$.

To illustrate the proof of Theorem 10.1, here we compute the product in the quantization $U_q(a)$ of a Lie bialgebra $a$ modulo $h^3$. In the text below we always assume summation over repeated indices.

Let $\{a_i, i \in I\}$ be a basis of $a$, and $\{b^i\}$ be the topological basis of $a^*$ dual to $\{a_i\}$. Let us write down the commutation relations for the Lie algebra $g = a \oplus a^*$:

(A1) $[a, a] = c_{ij}^k a_k, [b^i, b^j] = f_{ij}^k b^k, [a, b^i] = f_{ij}^k a_i - c_{ij}^k b^k$.

Let $1^+_{\gamma} \in M^+_\gamma$ be the functional on $M^+_\gamma$ defined by $1^+_{\gamma}(x 1_+^+) = \varepsilon(x), x \in U(a)$.

Let $\{M^+_\gamma\}$ be the filtration of $M^+_\gamma$ which was defined in Chapter 7.

For $x \in U(a)$, let $\psi_x : M_+ \rightarrow M^+_\gamma \oplus M_-$ be the $\gamma$-intertwiner such that

$\psi_x 1_+ = 1^+_{\gamma} \otimes x 1_-$ mod $(M^+_\gamma)$.

For $x, y \in U(a)$, we defined the quantized product $z = y \circ x$ to be the element of $U(a)[[h]]$ such that the operator $\psi_x$ is the composition

$M_+ \xrightarrow{\psi_x} M^+_\gamma \oplus M_-, \xrightarrow{1^+_{\gamma} \otimes x 1_- + c_{ij}^k a_k \otimes b^j} (M^+_\gamma \oplus M_-) \xrightarrow{\gamma^{-1}} M^+_\gamma \oplus M_-$

We want to compute the product $a_i \circ a_j$, modulo $h^3$. We fix elements $\rho_i \in M^+_\gamma, i \in I$, such that $\rho_i(1_+^+) = 0, \rho_i(b 1_+^+) = 0$. These elements are uniquely defined modulo $(M^+_\gamma)^2$.

Let $w^i \in \mathfrak{m}_-$ be the vectors such that

(A3) $\rho_i 1_- = 1^+_{\gamma} \otimes a_i 1_- + c_{ij}^k a_k \otimes b^j \mod (M^+_\gamma)^2 \oplus M_-$.

We must have $b^i \rho_i 1_- = 0$ for all $j$, so $1^+_{\gamma} \otimes b^i a_j 1_- + b^i \rho_i 1_, w^j = 0$. But $b^i \rho_i 1_+ = -\delta^i_1$, so we get $w^i = b^i a_i 1_- = -f^i_{ij} a_j 1_-$.

Etingof - Kazhdan Page 18
Thus we get
\[
\psi_{\alpha_1 \ldots \alpha_n 1} = 1^*_{\alpha_1 \ldots \alpha_n 1} - f_{1}^{\alpha_1} \rho_1 \otimes \alpha_1 \ldots \alpha_n 1 \pmod (M_{\alpha_1}^*)_2 \otimes M_{\alpha_n 1}.
\]

Using (A4), we get
\[
\psi_{\alpha_1 \alpha_2 \ldots \alpha_n 1} = (\alpha_1 \otimes 1 + 1 \otimes \alpha_2) \psi_{\alpha_1 \ldots \alpha_n 1} =
\]
\[
1^*_{\alpha_1 \alpha_2 \ldots \alpha_n 1} - f_{1}^{\alpha_1} \rho_1 \otimes \alpha_2 \ldots \alpha_n 1 - f_{1}^{\alpha_2} \rho_2 \otimes \alpha_1 \ldots \alpha_n 1 \pmod (M_{\alpha_1}^*)_2 \otimes M_{\alpha_n 1}.
\]

We have
\[
a_1 \rho_1 (b^1 1_+) = -\rho_1 (a_1 b^1 1_+) = \rho_1 (c_{11} b^1 1_+) = c_{11}^1.
\]

Thus, substituting (A6) into (A5), we get
\[
\psi_{\alpha_1 \alpha_2 \ldots \alpha_n 1} = 1^*_{\alpha_1 \alpha_2 \ldots \alpha_n 1} - f_{1}^{\alpha_1} \rho_1 \otimes \alpha_2 \ldots \alpha_n 1 - f_{1}^{\alpha_2} \rho_2 \otimes \alpha_1 \ldots \alpha_n 1 \pmod (M_{\alpha_2}^*)_2 \otimes M_{\alpha_n 1}.
\]

In particular, we have
\[
(1 \otimes \psi_{a_1}) \psi_{a_1 \alpha_2 \ldots \alpha_n 1} = 1^*_{a_1 a_2 \ldots \alpha_n 1} - f_{1}^{a_1} \rho_1 \otimes \alpha_2 \ldots \alpha_n 1 - f_{1}^{a_2} \rho_2 \otimes a_1 \ldots \alpha_n 1 + f_{1}^{a_2} \rho_2 \otimes \alpha_2 \ldots \alpha_n 1 \pmod (M_{a_1}^*)_2 \otimes M_{a_2 a_3 \ldots \alpha_n 1}.
\]

The definition of an associator implies
\[
\Phi = 1 + \frac{h^2}{24} [t_{11}, t_{22}] + O(h^3).
\]

(see [Dr2],[Dr3]). This means that the part of the $h^2$-coefficient of $\Phi^a_{b \otimes c}$ which belongs to $a^* \otimes a^* \otimes a$ is $\frac{1}{24} [c_{01} b \otimes b \otimes a_1^*].$

Now let us apply $\Phi^{-1}$ to both sides of (A8). We want to compute the answer in the form $1^*_{a_1 a_2 \ldots a \ldots} \otimes u \ldots \in M_{[a]}$. To do this, we only need to use the last two terms on the r.h.s. of (A8) and the $a^* \otimes a^* \otimes a$ part of the quadratic term of $\Phi$. The calculation gives
\[
\Phi^{-1} (1 \otimes \psi_{a_1}) \psi_{a_1 \alpha_2 \ldots \alpha_n 1} = 1^*_{a_1 \alpha_2 \ldots \alpha_n 1} \otimes u \pmod (M_{a_1}^*)_2 \otimes M_{\alpha_2 \ldots \alpha_n 1} + M_{a_1}^* \otimes (M_{\alpha_2}^*)_2 \otimes M_{\alpha_n 1},
\]
\[
u = a_1 a_2 \ldots a_n + \frac{h^2}{24} \left([f^1_{\rho_1} f^1_{\rho_2} c_{11}^1 a_1 a_2] + [f^1_{\rho_2} f^1_{\rho_1} c_{11}^1 a_2 a_1]\right).
\]

This shows that
\[
a_1 \circ a_2 = a_1 a_2 + \frac{h^2}{24} \left([f^1_{\rho_1} f^1_{\rho_2} c_{11}^1 a_1 a_2] + [f^1_{\rho_2} f^1_{\rho_1} c_{11}^1 a_2 a_1]\right) + O(h^3).
\]

This formula is analogous to the formula deduced by Deinvelt [Dr3] (equation 1.1).

It is easy to see that this formula contains only acyclic monomials. Therefore, this formula is universal.

Etingof - Kazhdan Page 19
REFERENCES