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SEIFERT SURFACES IN OPEN BOOKS, AND A NEW CODING ALGORITHM FOR LINKS

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ABSTRACT. We introduce a new standard form of a Seifert surface F . In that standard form, F is obtained by successively plumbing flat annuli to a disk D , where the gluing regions are all in D . We show that any link has a Seifert surface in the standard form, and thereby present a new way of coding a link. We present an algorithm to read the code directly from a braid presentation.

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1. INTRODUCTION

The trivial open book decomposition \mathcal{O} of the 3-sphere S^3 is a decomposition of S^3 into infinitely many disk (called pages) sharing their boundaries. An arc presentation of a link L is a presentation of L as a finite union of arcs each of which is properly embedded in a page. In [1], P. Cromwell introduced the concept of arc presentation and studied its properties.

It is well known that any link L admits a Seifert surface F (i.e., a compact, orientable, embedded surface whose boundary coincides with L). In this paper, we introduce a concept of positions of Seifert surfaces called the flat plumbing basket presentation, modifying the definition of arc presentation.

} will you have to increase the genus?

Definition 1.1. A Seifert surface is said to be a *flat plumbing basket* if it consists of a single page of \mathcal{O} and finitely many bands which are embedded in distinct pages. We say that a link L admits a *flat plumbing basket presentation* if there exists a flat plumbing basket F such that ∂F is equivalent to L . (See Figure 1.1.)

In this paper, we show that every link admits a flat plumbing basket presentation (Theorem 2.4 (1)).

} of what complexity?

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Q: Is there a set of moves on flat plumbing basket diagrams?
H's answer: Not yet, not a complete one

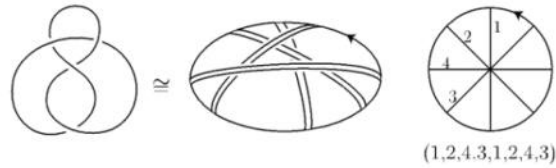


Figure 1.1: A flat plumbing basket presentation of the knot 4_1 .

Remark 1.2. (1) In [3], L. Rudolph called a Seifert surface F a *basket* if F is obtained from a disk D by successively plumbing unknotted (possibly twisted) annuli in such a way that every gluing region is embedded in D . It is obvious that a flat plumbing basket is obtained from a disk by plumbing flat annuli in such a way that every gluing region is embedded in the base disk. Hence a flat plumbing basket is a basket.

(2) In [2], C. Hayashi and M. Wada showed that any link admits a Seifert surface which is obtained from a disk D by successively plumbing flat annuli. Here we note that the gluing regions in their construction are not necessarily contained in D . In Section 4, we give an example showing a gap between their ‘flat annuli plumbing’ and our flat plumbing basket in terms of the minimal necessary number of plumbings to construct a specific link.

Definition 1.3. (1) A *flat basket diagram* \mathcal{D} is a pair consisting of a disk D (whose boundary has the counter-clockwise orientation) and a union of arcs a_1, \dots, a_n properly embedded in D with $\partial a_i \cap \partial a_j = \emptyset$ ($i \neq j$). We call the subscript i of a_i the *label* of the arc.

(2) For a flat basket diagram \mathcal{D} , by recording the labels of the arcs as one travels along ∂D , one obtains a word W in $\{1, \dots, n\}$ such that each letter appears exactly twice. We call W a *flat basket code* for \mathcal{D} .

Remark 1.4. (See Definition 2.1 for terminology.) Given a flat plumbing basket F , we can eventually obtain a disk by successively deplumbing flat annuli from F off the positive side of D . Each deplumbing corresponds to removing a band. By labelling the bands of F in the order of these deplumbings, we obtain a flat basket diagram (see Figure 1.1), and hence a flat basket code F . We also call that flat basket code a *flat basket code* for the link $L = \partial F$. Conversely, given a flat basket diagram \mathcal{D} , we can obtain the flat plumbing basket that \mathcal{D} presents by attaching flat bands b_n, b_{n-1}, \dots, b_1 on the positive side along a_n, a_{n-1}, \dots, a_1 in this order. Here we note that b_i is closer to D than b_{i+1} .

Example 1.5. The flat basket code $(1, 2, 4, 3, 1, 2, 4, 3)$ represents the figure-eight knot 4_1 (see Figure 1.1), and the code $(1, 2, 3, 4, 1, 2, 3, 4)$ represents the trefoil 3_1 . We see that any flat plumbing basket for 4_1 or 3_1 has at least four bands, since these two knots have genus 1, and any flat plumbing basket is compressible and hence not of minimal genus.

In Theorem 2.4 (2), we give an algorithm to obtain a flat basket code for any link L , using a braid presentation of L .

Remark 1.6. Let C_L be the set of flat basket codes for a link L . Each C_L is non-empty, and has a unique element c_L which has the shortest length and lexicographically minimal. Therefore, we can regard the set of all link types as a totally ordered set, by defining $L_i < L_j$ if and only if $c_{L_i} < c_{L_j}$.

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2. PRELIMINARIES AND THE STATEMENT OF THE RESULTS

First we recall the notion of *plumbing* and *deplumbing* of Seifert surfaces.

Definition 2.1. Let R be a Seifert surface in S^3 . We say that R is a plumbing of surfaces R_1 and $R_2 (\subset S^3)$ if the following conditions are satisfied:

- (1) $R = R_1 \cup R_2$ and $R_1 \cap R_2 = D$, where D is a square with four successive edges a_1, b_1, a_2, b_2 such that a_i (resp. b_i) is contained in ∂R_2 (resp. ∂R_1) and properly embedded in R_1 (resp. R_2).
- (2) There exist 3-balls $B_1, B_2 \subset S^3$ satisfying the following:
 - (2.1) $B_1 \cup B_2 = S^3$ and $B_1 \cap B_2 = \partial B_1 = \partial B_2$.
 - (2.2) $B_i \supset R_i$, $i = 1, 2$.
 - (2.3) $\partial B_1 \cap R_1 = \partial B_2 \cap R_2 = D$.

Here the square D is called the *gluing region* of the plumbing. We may say that R is obtained from R_1 by plumbing R_2 along D , or that R_1 is obtained from R by deplumbing R_2 .

Definition 2.2. Suppose a Seifert surface R is a plumbing of R_1 and R_2 along the gluing region D .

- (1) Let T be a subsurface of R_1 . We say that R_2 is *rooted in T* , if D is contained in T .
- (2) Since R_1 is oriented, it admits two sides, say the $+$ -side and $-$ -side. If the 3-ball B_2 lies in the ε -side ($\varepsilon = \pm$), we say that R is obtained from R_1 by plumbing R_2 along D on the ε -side. Conversely, we say that R_1 is obtained from R by deplumbing R_2 off the ε -side.

(3) Let a be an arc properly embedded in R_1 . Let N , be a regular neighborhood of a in R_1 . Push $N - \partial R_1$ into the ε -side slightly to make a band N' attached to R_1 . Let $R = R_1 \cup N'$. Then R is a plumbing of R_1 and a flat annulus (in fact, $N \cup N'$) on the ε -side. We say that R is obtained from R_1 by attaching a flat band along a on the ε -side.

Remark 2.3. (1) A Seifert surface F is isotopic to a flat plumbing basket if and only if F is obtained from a disk D by successively plumbing flat annuli each of which is rooted in D .

(2) In (1), some of the annuli may be plumbed on the $-$ side. However, by using isotopy as in Figure 2.1, we can regard all the annuli as plumbed on the $+$ side.

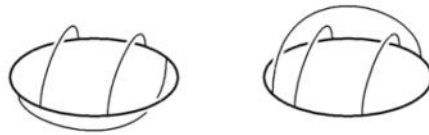


Figure 2.1: Flipping a band over.

Note that any link can be expressed as the closure of some n -braid [4], in particular, with a braid word of the form $\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1W$. (In fact, if L is the closure of a braid B , we may let $W = (\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1)^{-1}B$.) We adopt the following convention on braids.

Convention on braids. See Figure 2.2. In drawing braids, we put strings horizontally and call the top string the first string and so forth. Strings are oriented from left to right, and we read the braid word from the left end. The generator σ_i corresponds to a crossing of the i^{th} and $(i+1)^{\text{st}}$ strings, where the i^{th} string goes down crossing over the $(i+1)^{\text{st}}$ which goes up.

The main result of this paper is as follows:

Theorem 2.4. *Let L be an oriented link. Express L as a closed n -braid with a braid word $\sigma_{n-1}\sigma_{n-2}\cdots\sigma_1W$. Suppose the length of W is m and that W has s positive letters. Then we have the following:*

(1) *There exists a flat plumbing basket F with $m + 2s$ bands such that ∂F is isotopic to L .*

(2) *A flat basket code C (of length $2(m + 2s)$) coming from F is obtained by the following algorithm.*

Algorithm. *Let a_i be the i^{th} letter in W .*

Step 1: We give distinct labels in $\{1, \dots, m\}$ to a_i 's. The label $\ell(a_i)$ is defined to be k if a_i is in the k^{th} position with respect to the lexicographic order by the double index of a_i , where the double index of a_i is (q, i) if $a_i = \sigma_q^{\pm 1}$.

For each $i = 1, \dots, n$, let M_i denote the word obtained from the word $a_1 a_2 \dots a_m$ as follows:

M_1 is obtained by deleting all a_j 's which are not $\sigma_1^{\pm 1}$.

M_n is obtained by deleting all a_j 's which are not $\sigma_{n-1}^{\pm 1}$.

M_i ($1 < i < n$) is obtained by deleting all a_j 's which are not $\sigma_{i-1}^{\pm 1}$ or $\sigma_i^{\pm 1}$.

Let C_1 be the word obtained from the composed word $M_1 M_2 \dots M_n$ by replacing each a_j by $\ell(a_j)$. Note that C_1 is a word in $\{1, 2, \dots, m\}$ such that each letter appears exactly twice.

Step 2: Let $\{a_{p_1}, \dots, a_{p_s}\}$ ($p_1 < p_2 < \dots < p_s$) be the subset of $\{a_1, \dots, a_m\}$ such that each element corresponds to a positive letter. Then the desired code C is obtained as follows:

In C_1 , replace the second appearance of each $\ell(a_{p_i})$, $i = 1, \dots, s$, by a sequence $m + 2i, m + 2i - 1, \ell(a_{p_i}), m + 2i, m + 2i - 1$.

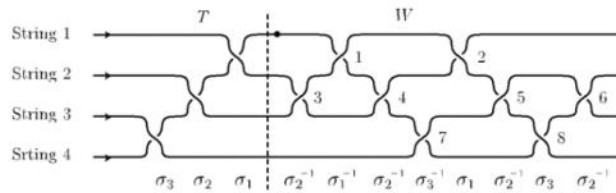


Figure 2.2: Labelling the letters in W .

Example 2.5. See Figure 2.2. Here, $W = \sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_3^{-1} \sigma_1 \sigma_2^{-1} \sigma_3 \sigma_2^{-1}$ and $m = 8, s = 2$. The double indices for a_1, \dots, a_8 are $(2, 1), (1, 2), (2, 3), (3, 4), (1, 5), (2, 6), (3, 7), (2, 8)$. Since $(1, 2) < (1, 5) < (2, 1) < (2, 3) < (2, 6) < (2, 8) < (3, 4) < (3, 7)$, we have: $(\ell(a_1), \ell(a_2), \ell(a_3), \ell(a_4), \ell(a_5), \ell(a_6), \ell(a_7), \ell(a_8)) = (3, 1, 4, 7, 2, 5, 8, 6)$. In Figure 2.2, numbers put beside the crossings present the labels. On the other hand, we have $M_1 = a_2 a_5, M_2 = a_1 a_2 a_3 a_5 a_6 a_8, M_3 = a_1 a_3 a_4 a_6 a_7 a_8, M_4 = a_4 a_7$. Hence we have: $C_1 = \ell(a_2) \ell(a_5) \cdot \ell(a_1) \ell(a_2) \ell(a_3) \ell(a_5) \ell(a_6) \ell(a_8) \cdot \ell(a_1) \ell(a_3) \ell(a_4) \ell(a_6) \ell(a_7) \ell(a_8) \cdot \ell(a_4) \ell(a_7) = (1\ 2\ 3\ 1\ 4\ 2\ 5\ 6\ 3\ 4\ 7\ 5\ 8\ 6\ 7\ 8)$. Since a_5 (with label 2) and a_7 (with label 8) correspond to the positive letters, we replace the second appearances of 2 and 8 in C_1 respectively by $(10\ 9\ 2\ 10\ 9)$ and

(12 11 8 12 11). Thus, we obtain a flat basket code

$$\mathcal{C} = (1\ 2\ 3\ 1\ 4\ 10\ 9\ 2\ 10\ 9\ 5\ 6\ 3\ 4\ 7\ 5\ 8\ 6\ 7\ 12\ 11\ 8\ 12\ 11).$$

Note that we can read off \mathcal{C}_1 , and hence \mathcal{C} from a geometric braid. See Figure 2.3, which is a schematic picture where the crossings corresponding to the letters of W are represented by arcs. The arcs are labelled from ‘top-left’ to ‘bottom-right’. In this picture, the closed braid B obtained by removing all the crossings corresponding to the letters in W is the unknot. We obtain the code \mathcal{C}_1 by travelling along B with the starting point depicted by the dot and recording the labels of the arcs as we pass their endpoints.

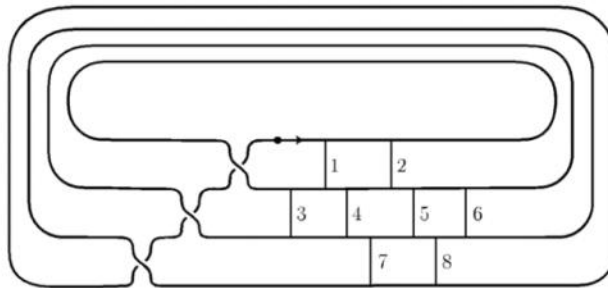


Figure 2.3: Reading a flat basket code.

3. PROOF OF THEOREM 2.4.

Suppose that L is the closure of an n -braid with a braid word TW , where T is $\sigma_{n-1} \cdots \sigma_2 \sigma_1$. Let W' be the word obtained from W by replacing each positive letter σ_i by $\sigma_i^{-1} \sigma_i \sigma_i$. Note that L is also presented by TW' . Let F be the Seifert surface for L obtained from the closure of the geometric braid presented by TW' by applying Seifert algorithm, i.e., F is obtained from n parallel horizontal disks D_1, \dots, D_n by attaching half twisted bands each corresponding to a letter in TW' , where D_i corresponds to the i^{th} string and, as we may suppose, the heights of D_1, \dots, D_n are in this order with D_1 the highest. We assume the face-side of D_1 coincides with the $+$ -side of F .

Let D be the disk in F consisting of D_1, \dots, D_n and the $n - 1$ bands corresponding to the letters in $T = \sigma_{n-1} \cdots \sigma_2 \sigma_1$. From now on, we regard F as consisting of the disk D and bands attached to D . Namely, T corresponds to the disk D , and W' corresponds to the bands attached to D .

Note that each band from W corresponds to some a_j . Then label the band with $\ell(a_j)$. Recall $\{a_{p_1}, \dots, a_{p_s}\}$ in Step 2. As we obtain F from the Seifert surface obtained from TW , we locally attach two bands near each band labelled $\ell(a_{p_i})$. See Figure 3.1. Deform two of the bands as specified in Figure 3.1 and (re)label the three bands as $m + 2i - 1, m + 2i, \ell(a_{p_i})$.

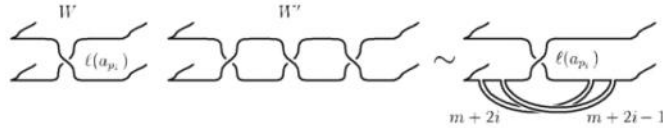


Figure 3.1: Deformation of bands in W' and relabeling.

Now there are $m + 2s$ bands attached to D . We call the first m bands (labelled from 1 to m) the *vertical bands*, since each of them connects two distinct sub-disks D_t and D_{t+1} of D . On the other hand, we call the last $2s$ bands (labelled from $m + 1$ to $m + 2s$) the *horizontal bands* since each of them connects a single sub-disk of D . Note that all the vertical bands are twisted in the same way. Actually, the system of vertical bands corresponds to the word obtained from W by replacing all σ_* by σ_*^{-1} . We denote this Seifert surface for L by F' and regard the disk D as a subsurface in F' .

We prove the following:

Claim 3.1. *The surface F' thus obtained is isotopic to a flat plumbing basket for L , where all bands are rooted in D .*

We show that we can deplumb $m + 2s$ flat annuli, say, $A_1, A_2, \dots, A_{m+2s}$ each rooted in D successively in this order, in such a way that deplumbing A_i corresponds to removing the band labelled i .

Let $h_1, h_2, \dots, h_{n-2}, h_{n-1} = m$ be integers such that (i) there exist h_1 vertical bands connecting D_1 and D_2 , (ii) there exist $h_i - h_{i-1}$ ($2 \leq i \leq n - 1$) vertical bands connecting D_i and D_{i+1} .

See Figure 3.2 for the same braid used in Example 2.5, where the bands connecting the subdisks of the base disk D are deformed so that only one side of D is visible. Let F_1 be the surface obtained from F' by removing the bands labelled $1, \dots, h_1$. It is directly observed from Figure 3.2 that F_1 is obtained from F' by deplumbing a sequence of flat annuli, say, A_1, \dots, A_{h_1} (where A_j corresponds to the band j) each rooted in D . Here, each of the gluing regions (represented by dotted lines) goes once though the band in D connecting D_1 and D_2 . Then repeat the same procedure for the bands labelled i ($h_1 < i \leq h_2$) to obtain a new surface F_2 and so forth.

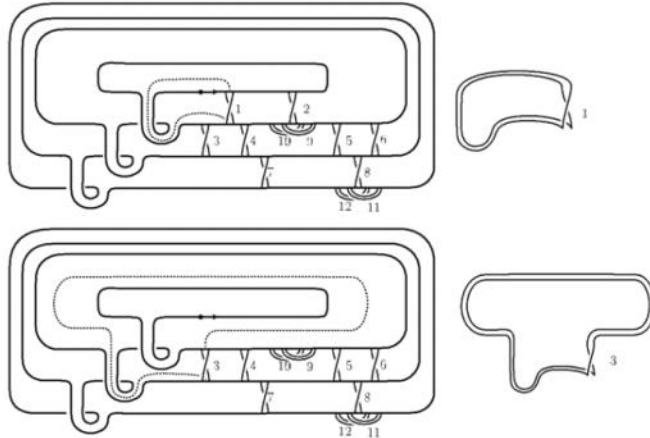


Figure 3.2: Removing vertical bands by deplumbing flat annuli.

Finally we obtain a surface F_{n-1} which is a union of D and the horizontal bands. We can remove the $2s$ horizontal bands by deplumbing flat annuli rooted in D , say, A_{m+1}, \dots, A_{m+2s} in this order, where A_j corresponds to the band j . Finally, we have constructed a Seifert surface for L which is isotopic to a flat plumbing basket with $m+2s$ bands. The proof of Theorem 2.4 (1) is now completed.

For the proof of Theorem 2.4 (2), it is enough to confirm that a flat basket code of F' obtained by using A_1, \dots, A_{m+2s} coincides with the code that the algorithm yields. This can be done by travelling along ∂D starting from the dot indicated in Figure 2.3.

4. EXAMPLES

In [2], Hayashi and Wada showed that any oriented link admits a Seifert surface which is obtained from a disk by plumbing flat annuli. A flat plumbing basket can be regarded as such an example, but not vice versa. In this section, we exhibit a Seifert surface which is obtained from a disk by successively plumbing flat annuli, but which is not isotopic to any flat plumbing basket.

Proposition 4.1. *Let L be the 2-bridge link $S(4, 1)$, oriented so that L spans an unknotted annulus with two full-twists (Figure 4.1). Then we have the following.*

(1) L admits a Seifert surface which is obtained from a disk by plumbing

three flat annuli.

(2) Any flat plumbing basket for L has at least 5 bands.

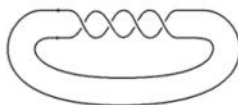


Figure 4.1: The oriented 2-bridge link $S(4, 1)$.

Proof. Figure 4.2 shows Proposition 4.1 (1): Consider a dotted line d in a surface obtained by plumbing two flat annuli to a disk, as in Figure 4.2 (a). Then by plumbing another flat annulus along d on the face-side, we obtain a Seifert surface isotopic to that in Figure 4.2 (b), whose boundary is equivalent to L . Note that the gluing region for the third plumbing goes through a previously plumbed flat annulus.

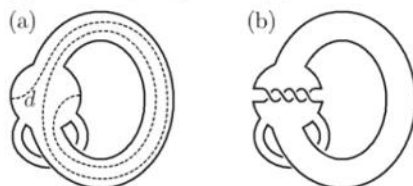


Figure 4.2: A flat annuli plumbing for $S(4, 1)$ with three bands

To prove (2), we examine flat plumbing baskets which have ≤ 4 bands. Suppose a flat plumbing basket F has n bands. If $n \leq 2$, then it is easy to observe that ∂F is either the trivial knot or a trivial link with two or three components. If $n = 3$, then there are four possible patterns of ambient arcs of flat basket diagrams depicted in Figure 4.3.

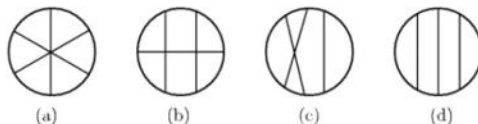


Figure 4.3: Flat basket diagrams with three arcs.

In Figure 4.3 (a), ∂F is a (positive or negative) Hopf link. In other cases, ∂F is a trivial link with two or four components. If $n = 4$, then $\partial F \neq L$ since ∂F has an odd number of connected components. (Note that removing a band changes the number of connected components by ± 1). Hence Proposition 4.1 is proved. \square

Note that $S(4, 1)$ admits a flat plumbing basket presentation with five bands as in Figure 4.4, with a flat basket code $(1, 2, 3, 4, 5, 1, 4, 5, 2, 3)$.

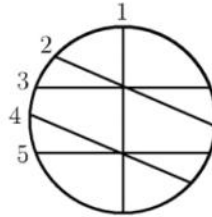


Figure 4.4: A flat plumbing basket for $S(4, 1)$ with five bands.

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