Pensieve header: Alexander from Heisenberg games, continues pensieve://Talks/LearningSeminarOnCategorification-2006/ .

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\2020-07"]
```

Out[∘]=
```
C:\drorbn\AcademicPensieve\2020-07
```

tex

\def\cellscale{0.78}
\def\nbpdfInput#1{\vskip 1mm\par\noindent\includegraphics[scale=\cellscale]{#1}}
\def\nbpdfEcho#1{\vskip 1mm\par\noindent\includegraphics[scale=\cellscale]{#1}}
\def\nbpdfOutput#1{\vskip 1mm\par\noindent\includegraphics[scale=\cellscale]{#1}}
\def\nbpdfSubsection#1{\vskip 1mm\par\noindent\includegraphics[scale=\cellscale]{#1}}

pdf

In[∘]:=
```
CF = ExpandNumerator@*ExpandDenominator@*PowerExpand@*Factor;
```

In[∘]:=
```
𝔼_s1_[ωQ1__] ≡ 𝔼_s2_[ωQ2__] := s1 === s2 ∧ Simplify[{ωQ1} == {ωQ2}]
```

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In[∘]:=
```
𝔼_{A1_→B1_}[ω1_, Q1_] 𝔼_{A2_→B2_}[ω2_, Q2_] ^:= 𝔼_{A1⋃A2→B1⋃B2}[ω1 ω2, Q1 + Q2]
```

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In[∘]:=
```
(𝔼_{A1_→B1_}[ω1_, Q1_] // 𝔼_{A2_→B2_}[ω2_, Q2_]) /; (B1* === A2) :=
 Module[{i, j, E1, F1, G1, E2, F2, G2, I, M = Table},
  I = IdentityMatrix@Length@B1;
  E1 = M[∂_{i,j}Q1, {i, A1}, {j, B1}]; E2 = M[∂_{i,j}Q2, {i, A2}, {j, B2}];
  F1 = M[∂_{i,j}Q1, {i, A1}, {j, A1}]; F2 = M[∂_{i,j}Q2, {i, A2}, {j, A2}];
  G1 = M[∂_{i,j}Q1, {i, B1}, {j, B1}]; G2 = M[∂_{i,j}Q2, {i, B2}, {j, B2}];
  𝔼_{A1→B2}[CF[ω1 ω2 Det[I - F2.G1]^(1/2)], CF@Plus[
     If[A1 === {} ∨ B2 === {}, 0, A1.E1.Inverse[I - F2.G1].E2.B2],
     If[A1 === {}, 0, 1/2 A1.(F1 + E1.F2.Inverse[I - G1.F2].E1^T).A1],
     If[B2 === {}, 0, 1/2 B2.(G2 + E2^T.G1.Inverse[I - F2.G1].E2).B2]]]]
```

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In[∘]:=
```
A_ \ B_ := Complement[A, B];
(𝔼_{A1_→B1_}[ω1_, Q1_] // 𝔼_{A2_→B2_}[ω2_, Q2_]) /; (B1* =!= A2) :=
 𝔼_{A1⋃(A2\B1*)→B1⋃A2*}[ω1, Q1 + Sum[ζ* ζ, {ζ, A2 \ B1*}]] //
  𝔼_{B1*⋃A2→B2⋃(B1\A2*)}[ω2, Q2 + Sum[z* z, {z, B1 \ A2*}]]
```

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In[∘]:=
```
{p*, x*, π*, ξ*} = {π, ξ, p, x}; (u_i_)* := (u*)_i; l_List* := #* & /@ l;
```

A proof of the formula for R is at http://drorbn.net/cat20.

pdf

In[∘]:=
$$R_{i\_,j\_} := \mathbb{E}_{\{\}\to\{p_i,x_i,p_j,x_j\}}\left[T^{-1/2},\ \mu\,(p_i - p_j)\,x_j\right];$$
$$\overline{R}_{i\_,j\_} := \mathbb{E}_{\{\}\to\{p_i,x_i,p_j,x_j\}}\left[T^{1/2},\ \frac{\mu}{1+\mu}\,(p_j - p_i)\,x_j\right];$$
$$C_{i\_} := \mathbb{E}_{\{\}\to\{p_i,x_i\}}\left[T^{-1/2},\ 0\right];\ \ \overline{C}_{i\_} := \mathbb{E}_{\{\}\to\{p_i,x_i\}}\left[T^{1/2},\ 0\right];$$

In[∘]:= C_1

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1}{\sqrt{T}},\ 0\right]$

In[∘]:= $\{(R_{1,2})_h,\ (\overline{R}_{1,2})_h\}$

Out[∘]=
$$\left\{\left(\begin{array}{ccc} \frac{1}{\sqrt{T}} & x_1 & x_2 \\ p_1 & 0 & \mu \\ p_2 & 0 & -\mu \end{array}\right)_h,\ \left(\begin{array}{ccc} \sqrt{T} & x_1 & x_2 \\ p_1 & 0 & -\frac{\mu}{1+\mu} \\ p_2 & 0 & \frac{\mu}{1+\mu} \end{array}\right)_h\right\}$$

In[∘]:= $R_{1,2} \equiv \mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2\}}\left[T^{-1/2},\ \{x_1, x_2\}.\begin{pmatrix} 0 & 0 \\ T-1 & 1-T \end{pmatrix}.\{p_1, p_2\}\right]$

Out[∘]= $\{0,\ (1 - T + \mu)\ (p_1 - p_2)\ x_2\} == \{0, 0\}$

A proof of the formula for hm is at http://drorbn.net/cat20.

pdf

In[∘]:=
$$hm_{i\_,j\_\to k\_} := \mathbb{E}_{\{\pi_i,\xi_i,\pi_j,\xi_j\}\to\{p_k,x_k\}}\left[1,\ -\xi_i\,\pi_j + (\pi_i + \pi_j)\,p_k + (\xi_i + \xi_j)\,x_k\right]$$

In[∘]:= $hm_{1,2\to3}$

Out[∘]= $\mathbb{E}_{\{\pi_1,\xi_1,\pi_2,\xi_2\}\to\{p_3,x_3\}}\left[1,\ p_3\ (\pi_1 + \pi_2) - \pi_2\,\xi_1 + x_3\ (\xi_1 + \xi_2)\right]$

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In[∘]:=
```
𝔼{}→vs_[ωi_, Q_]h := Module[{ps, xs, M},
  ps = Cases[vs, p_]; xs = Cases[vs, x_];
  M = Table[ωi, 1 + Length@ps, 1 + Length@xs];
  M[[2 ;;, 2 ;;]] = Table[CF[∂i,j Q], {i, ps}, {j, xs}];
  M[[2 ;;, 1]] = ps; M[[1, 2 ;;]] = xs;
  MatrixForm[M]h]
```

tex

\parpic[r]{\scalebox{1}{\input{R3.pdftex_t}}}
{\red\bf Proof of Reidemeister 3.}
{\def\nbpdfOutput#1{\vskip
1mm\par\noindent\includegraphics[scale=\cellscale]{#1}\hfill\text{$\Box$\quad}}

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In[∘]:= $(R_{1,2}\,R_{4,3}\,R_{5,6}\ //\ hm_{1,4\to1}\,hm_{2,5\to2}\,hm_{3,6\to3}) == (R_{2,3}\,R_{1,6}\,R_{4,5}\ //\ hm_{1,4\to1}\,hm_{2,5\to2}\,hm_{3,6\to3})$

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Out[∘]= True

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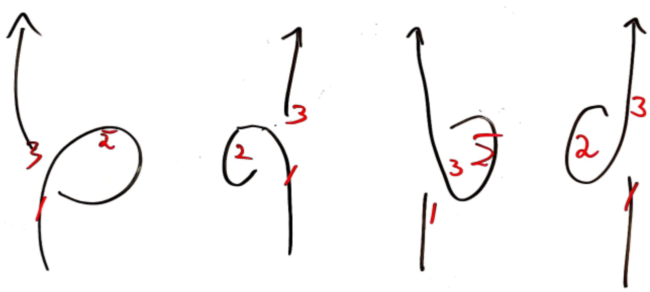
}

Reidemeister 2.

In[∘]:= $\{\overline{R}_{1,2}\,R_{3,4}\ //\ hm_{1,3\to1}\,hm_{2,4\to2},\ \overline{R}_{1,4}\,R_{3,2}\ //\ hm_{1,3\to1}\,hm_{2,4\to2}\}$

Out[∘]= $\{\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}[1,0],\ \mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}[1,0]\}$

Reidemeister 1's.

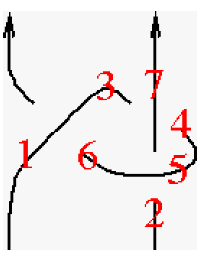


In[∘]:= $\{(R_{1,3}\,\overline{C}_2)\ //\ hm_{1,2\to1}\ //\ hm_{1,3\to1},\ (\overline{R}_{1,3}\,C_2)\ //\ hm_{1,2\to1}\ //\ hm_{1,3\to1},$
$(\overline{R}_{3,1}\,\overline{C}_2)\ //\ hm_{1,2\to1}\ //\ hm_{1,3\to1},\ (R_{3,1}\,C_2)\ //\ hm_{1,2\to1}\ //\ hm_{1,3\to1}\}$

Out[∘]= $\left\{\mathbb{E}_{\{\}\to\{p_1,x_1\}}[1,0],\ \mathbb{E}_{\{\}\to\{p_1,x_1\}}[1,0],\ \mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{T}{1+\mu},0\right],\ \mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1+\mu}{T},0\right]\right\}$

tex

{\red\bf The ``First Tangle''.}



pdf

In[∘]:= $\text{Factor}\ /@\ (z = R_{1,6}\,\overline{C}_3\,\overline{R}_{7,4}\,\overline{R}_{5,2}\ //\ hm_{1,3\to1}\ //\ hm_{1,4\to1}\ //\ hm_{1,5\to1}\ //\ hm_{1,6\to1}\ //\ hm_{2,7\to2})$

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,p_2,x_1,x_2\}}\left[\frac{T\,(1+2\,\mu)}{(1+\mu)^2},\ \frac{\mu\,(p_1-p_2)\,(x_1+\mu\,x_1-x_2)}{1+2\,\mu}\right]$

tex

\vskip -3mm

\parpic[r]{\scalebox{1}{\input{FirstTangle.pdftex_t}}}

pdf

In[∘]:= z_h

pdf

Out[∘]= $\begin{pmatrix} \frac{T+2\,T\,\mu}{1+2\,\mu+\mu^2} & x_1 & x_2 \\ p_1 & \frac{\mu+\mu^2}{1+2\,\mu} & -\frac{\mu}{1+2\,\mu} \\ p_2 & \frac{-\mu-\mu^2}{1+2\,\mu} & \frac{\mu}{1+2\,\mu} \end{pmatrix}_h$

tex

\parpic[r]{\scalebox{0.8}{\input{817.pdftex_t}}}

{\red\bf The knot $8_{17}$$.}

pdf

In[∘]:=
```
z = R̄12,1 R̄27 R̄83 R̄4,11 R16,5 R6,13 R14,9 R10,15;
Table[z = z // hm1k→1, {k, 2, 16}] // Last
```

pdf

Out[∘]= $\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{-1-3\mu-2\mu^2+\mu^3+3\mu^4+2\mu^5+\mu^6}{1+3\mu+3\mu^2+\mu^3}, 0\right]$

tex

{\red\bf Proof of Theorem 3, (3).}
{\def\nbpdfOutput#1{\vskip
1mm\par\noindent\includegraphics[scale=\cellscale]{#1}\hfill\text{$\Box$\quad}}

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In[∘]:= $\left\{\left(\gamma 1 = \mathbb{E}_{\{\}\to\{p_1,x_1,p_2,x_2,p_3,x_3\}}\left[\omega, \{p_1, p_2, p_3\}.\begin{pmatrix}\alpha & \beta & \theta\\ \gamma & \delta & \epsilon\\ \phi & \psi & \Xi\end{pmatrix}.\{x_1, x_2, x_3\}\right]\right)_h, (\gamma 1 \;//\; hm_{1,2\to 0})_h\right\}$

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Out[∘]= $\left\{\begin{pmatrix}\omega & x_1 & x_2 & x_3\\ p_1 & \alpha & \beta & \theta\\ p_2 & \gamma & \delta & \epsilon\\ p_3 & \phi & \psi & \Xi\end{pmatrix}_h, \begin{pmatrix}\omega+\gamma\omega & x_0 & x_3\\ p_0 & \frac{\alpha+\beta+\gamma+\beta\gamma+\delta-\alpha\delta}{1+\gamma} & \frac{\epsilon-\alpha\epsilon+\theta+\gamma\theta}{1+\gamma}\\ p_3 & \frac{\phi-\delta\phi+\psi+\gamma\psi}{1+\gamma} & \frac{\Xi+\gamma\Xi-\epsilon\phi}{1+\gamma}\end{pmatrix}_h\right\}$

tex

}

In[∘]:= $\text{Simplify}\left[\text{Table}\left[\partial_{i,j}\left(\gamma 1 \;//\; hm_{1,2\to 0}\right)[[2]], \{i, \{p_0, p_3\}\}, \{j, \{x_0, x_3\}\}\right] == \begin{pmatrix}1+\beta-\frac{(1-\alpha)(1-\delta)}{1+\gamma} & \theta+\frac{(1-\alpha)\epsilon}{1+\gamma}\\ \psi+\frac{(1-\delta)\phi}{1+\gamma} & \Xi-\frac{\epsilon\phi}{1+\gamma}\end{pmatrix}\right]$

Out[∘]= True

In[∘]:=
```
MatrixForm@Simplify[
  IdentityMatrix[2] - Table[∂i,j (γ1 // hm1,2→0)[[2]], {i, {p0, p3}}, {j, {x0, x3}}] /.
   Thread[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ} -> {1 - α, -β, -γ, 1 - δ, -θ, -ϵ, -ϕ, -ψ, 1 - Ξ}]
 ]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix}\beta-\frac{\alpha\delta}{-1+\gamma} & -\frac{\alpha\epsilon}{-1+\gamma}+\theta\\ -\frac{\delta\phi}{-1+\gamma}+\psi & \Xi-\frac{\epsilon\phi}{-1+\gamma}\end{pmatrix}$$

Recycling

Our PBW ordering is $\{p, x\}$.

We are at $[p, x] = 1$ and $R = e^{-t\otimes px + tp\otimes x}$. Let $a = p\,x$. Then $[a, x] = x$.

Claim. $e^{-t\,a + tp\,x} = e^{-t\,a}\, e^{t^{-1}(e^t-1)\,tp\,x}$.

Proof. Use a 2d representation:

In[∘]:= $\rho a = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix};\ \rho x = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix};\ \rho a.\rho x - \rho x.\rho a == \rho x$

Out[∘]= True

In[ ]:=
```
Simplify[MatrixExp[-t ρa + tp ρx] == MatrixExp[-t ρa].MatrixExp[t^-1 (e^t - 1) tp ρx]]
```
Out[ ]= True

Claim. $e^{t p x} = \mathbb{O}\left[e^{(1-e^{-t}) p x}\right]$.

Proof. True at $t = 0$, test ∂_t using $p\, x\, \mathbb{O}[f] = \mathbb{O}[p(x f - \partial_p f)]$:

In[ ]:=
```
Simplify[p (x e^((1-e^-t) p x) - ∂_p e^((1-e^-t) p x)) == ∂_t e^((1-e^-t) p x)]
```
Out[ ]= True

Claim. $\mathbb{O}[e^{-t a + \text{tp}\, x}] = e^{(e^t - 1) p x + t^{-1}(e^t - 1)\,\text{tp}\, x}$

In[ ]:=
```
Collect[(1 - e^-t) p x + t^-1 (e^-t - 1) tp x, {tp}, Simplify]
```
Out[ ]= $\left(1 - e^{-t}\right) p\, x + \frac{\left(-1 + e^{-t}\right) \text{tp}\, x}{t}$

The Trefoil

In[ ]:=
```
Z31 = R_{1,5} R_{6,2} R_{3,7} C̄_4;
Do[Z31 = Z31 // hm_{1,r→1}, {r, 2, 7}];
Simplify /@ Z31
```
Out[ ]= $\mathbb{E}_{\{\}\to\{p_1,x_1\}}\left[\frac{1 + \mu + \mu^2}{T}, 0\right]$