Pensieve header: Implementing and verifying gl_n^ϵ in its “cyclic” presentation. Continues pensieve://2017-02/ (though in different directions).

In[•]:=
```
n = 5;
```

In[•]:=
```
χ_cond_ := If[TrueQ@cond, 1, 0];
B[0, _] = 0; B[_, 0] = 0;
B[c_ * x : (x | a | b)__, y_] := Expand[c B[x, y]];
B[y_, c_ * x : (x | a | b)__] := Expand[c B[y, x]];
B[x_Plus, y_] := B[#, y] & /@ x;
B[x_, y_Plus] := B[x, #] & /@ y;
P[0, _] = 0; P[_, 0] = 0;
P[c_ * x : (x | a | b)__, y_] := Expand[c P[x, y]];
P[y_, c_ * x : (x | a | b)__] := Expand[c P[y, x]];
P[x_Plus, y_] := P[#, y] & /@ x;
P[x_, y_Plus] := P[x, #] & /@ y;
```

In[•]:=
```
P[x_{i_,j_}, x_{k_,l_}] := χ_{j==k∧i==l};
P[x__, (a | b)_] = 0; P[(a | b)_, x__] = 0;
P[a_{i_}, b_{j_}] := 2 χ_{i==j}; P[b_{j_}, a_{i_}] := 2 χ_{i==j};
P[a_, a_] = 0; P[b_, b_] = 0;
```

In[•]:=
```
λ[x_{i_,j_}] := Mod[j - i, n];
x_{i_,i_} := b_i / 2 + ϵ a_i / 2
```

In[•]:=
```
B[(a | b)_, (a | b)_] = 0;

B[x_{i_,j_}, x_{k_,l_}] := ϵ^{χ_{λ[x_{i,j}]+λ[x_{k,l}]>n}} (χ_{j==k} x_{i,l} - χ_{l==i} x_{k,j});

B[a_{i_}, x_{j_,k_}] := (χ_{i==j} - χ_{i==k}) x_{j,k};
B[b_{i_}, x_{j_,k_}] := ϵ (χ_{i==j} - χ_{i==k}) x_{j,k};
```

In[•]:=
```
B[y_, x_] := Expand[-B[x, y]];
```

In[•]:=
```
cyc[k_Integer] := Mod[k + 1, n, 1];
Cyc[ℰ_] := ℰ /. {x_{i_,j_} :> x_{cyc@i,cyc@j}, a_{i_} :> a_{cyc@i}, b_{i_} :> b_{cyc@i}}
```

In[•]:=
```
Act_{σ_List}[ℰ_] := ℰ /. {x_{i_,j_} :> x_{σ⟦i⟧,σ⟦j⟧}, a_{i_} :> a_{σ⟦i⟧}, b_{i_} :> b_{σ⟦i⟧}}
```

In[•]:=
```
ϕ_λ[ℰ_] := ℰ /. {x_{i_,j_} /; i > j :> λ x_{i,j}, b_{i_} :> λ b_i};
```

In[•]:=
```
Basis[n_] := Sort@Flatten[
    {Table[x_{i,j}, {i, n}, {j, Delete[Range[n], i]}], Table[a_i, {i, n}], Table[b_i, {i, n}]}];
```

In[•]:=
```
Basis[4]
```
Out[•]=
$$\{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, x_{1,2}, x_{1,3}, x_{1,4}, x_{2,1}, x_{2,3}, x_{2,4}, x_{3,1}, x_{3,2}, x_{3,4}, x_{4,1}, x_{4,2}, x_{4,3}\}$$

In[•]:=
```
n = 2;
Table[
  {u, v} → B[u, v],
  {u, Basis[n]}, {v, Basis[n]}
 ] // MatrixForm
```
Out[•]//MatrixForm=
$$\begin{pmatrix} \{a_1, a_1\} \to 0 & \{a_1, a_2\} \to 0 & \{a_1, b_1\} \to 0 & \{a_1, b_2\} \to 0 & \{a_1, x_{1,2}\} \to \\ \{a_2, a_1\} \to 0 & \{a_2, a_2\} \to 0 & \{a_2, b_1\} \to 0 & \{a_2, b_2\} \to 0 & \{a_2, x_{1,2}\} \to \\ \{b_1, a_1\} \to 0 & \{b_1, a_2\} \to 0 & \{b_1, b_1\} \to 0 & \{b_1, b_2\} \to 0 & \{b_1, x_{1,2}\} \to \\ \{b_2, a_1\} \to 0 & \{b_2, a_2\} \to 0 & \{b_2, b_1\} \to 0 & \{b_2, b_2\} \to 0 & \{b_2, x_{1,2}\} \to - \\ \{x_{1,2}, a_1\} \to -x_{1,2} & \{x_{1,2}, a_2\} \to x_{1,2} & \{x_{1,2}, b_1\} \to -\in x_{1,2} & \{x_{1,2}, b_2\} \to \in x_{1,2} & \{x_{1,2}, x_{1,2}\} \\ \{x_{2,1}, a_1\} \to x_{2,1} & \{x_{2,1}, a_2\} \to -x_{2,1} & \{x_{2,1}, b_1\} \to \in x_{2,1} & \{x_{2,1}, b_2\} \to -\in x_{2,1} & \{x_{2,1}, x_{1,2}\} \to -\frac{\in a_1}{2} + \end{pmatrix}$$

In[•]:=
```
n = 4;
Union@Table[
  ({u, v} = t); B[u, v] + B[v, u],
  {t, Tuples[Basis[n], 2]}
 ]
```
Out[•]=
$$\{0\}$$

In[•]:=
```
n = 4;
DeleteCases[Table[
  ({u, v, w} = t) → B[u, B[v, w]] + B[v, B[w, u]] + B[w, B[u, v]],
  {t, Tuples[Basis[n], 3]}
 ], _ → 0]
```
Out[•]=
$$\{\}$$

In[•]:=
```
n = 4;
DeleteCases[Flatten@Table[
   {u, v, w} → P[B[u, v], w] + P[v, B[u, w]],
   {u, Basis[n]}, {v, Basis[n]}, {w, Basis[n]}
  ], _ → 0]
```
Out[•]=
$$\{\}$$

In[•]:=
```
(# → Cyc[#]) & /@ Basis[4]
```
Out[•]=
$$\{a_1 \to a_2, a_2 \to a_3, a_3 \to a_4, a_4 \to a_1, b_1 \to b_2, b_2 \to b_3, b_3 \to b_4, b_4 \to b_1,$$
$$x_{1,2} \to x_{2,3}, x_{1,3} \to x_{2,4}, x_{1,4} \to x_{2,1}, x_{2,1} \to x_{3,2}, x_{2,3} \to x_{3,4}, x_{2,4} \to x_{3,1},$$
$$x_{3,1} \to x_{4,2}, x_{3,2} \to x_{4,3}, x_{3,4} \to x_{4,1}, x_{4,1} \to x_{1,2}, x_{4,2} \to x_{1,3}, x_{4,3} \to x_{1,4}\}$$

In[•]:=
```
n = 4;
DeleteCases[Flatten@Table[
   {u, v} → Cyc[B[u, v]] - B[Cyc[u], Cyc[v]],
   {u, Basis[n]}, {v, Basis[n]}
  ], _ → 0]
```
Out[•]=
$$\{\}$$

```
In[•]:= n = 4;
        DeleteCases[Flatten@Table[
            {u, v} → Cyc[P[u, v]] - P[Cyc[u], Cyc[v]],
            {u, Basis[n]}, {v, Basis[n]}
          ], _ → 0]
```

Out[•]= {}

```
In[•]:= n = 5;
        Select[Permutations[Range[n]],
         σ ↦ And @@ Flatten[Table[
             Act_σ[B[u, v]] === B[Act_σ[u], Act_σ[v]],
             {u, Basis[n]}, {v, Basis[n]}
            ]]
        ]
```

Out[•]= {{1, 2, 3, 4, 5}, {2, 3, 4, 5, 1}, {3, 4, 5, 1, 2}, {4, 5, 1, 2, 3}, {5, 1, 2, 3, 4}}

```
In[•]:= n = 4;
        Block[{ϵ = 1}, Select[Permutations[Range[n]],
          σ ↦ And @@ Flatten[Table[
              Act_σ[B[u, v]] === B[Act_σ[u], Act_σ[v]],
              {u, Basis[n]}, {v, Basis[n]}
             ]]
         ]]
```

Out[•]= {{1, 2, 3, 4}, {1, 2, 4, 3}, {1, 3, 2, 4}, {1, 3, 4, 2}, {1, 4, 2, 3}, {1, 4, 3, 2},
{2, 1, 3, 4}, {2, 1, 4, 3}, {2, 3, 1, 4}, {2, 3, 4, 1}, {2, 4, 1, 3}, {2, 4, 3, 1},
{3, 1, 2, 4}, {3, 1, 4, 2}, {3, 2, 1, 4}, {3, 2, 4, 1}, {3, 4, 1, 2}, {3, 4, 2, 1},
{4, 1, 2, 3}, {4, 1, 3, 2}, {4, 2, 1, 3}, {4, 2, 3, 1}, {4, 3, 1, 2}, {4, 3, 2, 1}}

```
In[•]:= n = 4;
        Union@Flatten@Table[
            (B[u, v] /. ϵ → 1) == ϕ_ϵ@B[ϕ_{1/ϵ}@u, ϕ_{1/ϵ}@v],
            {u, Basis[n]}, {v, Basis[n]}
          ]
```

Out[•]= {True}

```
In[•]:= n = 4; Table[u → ϕ_ϵ@Act_{2,3,4,1}@ϕ_{1/ϵ}@u, {u, Basis[n]}]
```

$$\text{Out[•]= }\left\{a_1 \to a_2,\ a_2 \to a_3,\ a_3 \to a_4,\ a_4 \to a_1,\ b_1 \to b_2,\ b_2 \to b_3,\ b_3 \to b_4,\ b_4 \to b_1,\ x_{1,2} \to x_{2,3},\ x_{1,3} \to x_{2,4},\ x_{1,4} \to \in x_{2,1},\ x_{2,1} \to x_{3,2},\ x_{2,3} \to x_{3,4},\ x_{2,4} \to \in x_{3,1},\ x_{3,1} \to x_{4,2},\ x_{3,2} \to x_{4,3},\ x_{3,4} \to \in x_{4,1},\ x_{4,1} \to \frac{x_{1,2}}{\in},\ x_{4,2} \to \frac{x_{1,3}}{\in},\ x_{4,3} \to \frac{x_{1,4}}{\in}\right\}$$

```
In[•]:= n = 4; Table[u → ϕ_{1/ϵ}@Act_{2,3,4,1}@ϕ_ϵ@u, {u, Basis[n]}]
```

$$\text{Out[•]= }\left\{a_1 \to a_2,\ a_2 \to a_3,\ a_3 \to a_4,\ a_4 \to a_1,\ b_1 \to b_2,\ b_2 \to b_3,\ b_3 \to b_4,\ b_4 \to b_1,\ x_{1,2} \to x_{2,3},\ x_{1,3} \to x_{2,4},\ x_{1,4} \to \frac{x_{2,1}}{\in},\ x_{2,1} \to x_{3,2},\ x_{2,3} \to x_{3,4},\ x_{2,4} \to \frac{x_{3,1}}{\in},\ x_{3,1} \to x_{4,2},\ x_{3,2} \to x_{4,3},\ x_{3,4} \to \frac{x_{4,1}}{\in},\ x_{4,1} \to \in x_{1,2},\ x_{4,2} \to \in x_{1,3},\ x_{4,3} \to \in x_{1,4}\right\}$$

In[•]:=
```
n = 3;
MatrixForm@Table[u → ϕ_ϵ@Act_σ@ϕ_{1/ϵ}@u, {σ, Permutations@Range@n}, {u, Basis[n]} ]
```

Out[•]//MatrixForm=

$$\begin{pmatrix}
a_1\to a_1 & a_2\to a_2 & a_3\to a_3 & b_1\to b_1 & b_2\to b_2 & b_3\to b_3 & x_{1,2}\to x_{1,2} & x_{1,3}\to x_{1,3} & x_{2,1}\to x_{2,1} & x_{2,3}\to x_{2,3}\\
a_1\to a_1 & a_2\to a_3 & a_3\to a_2 & b_1\to b_1 & b_2\to b_3 & b_3\to b_2 & x_{1,2}\to x_{1,3} & x_{1,3}\to x_{1,2} & x_{2,1}\to x_{3,1} & x_{2,3}\to \in x_{3,2}\\
a_1\to a_2 & a_2\to a_1 & a_3\to a_3 & b_1\to b_2 & b_2\to b_1 & b_3\to b_3 & x_{1,2}\to \in x_{2,1} & x_{1,3}\to x_{2,3} & x_{2,1}\to \frac{x_{1,2}}{\in} & x_{2,3}\to x_{1,3}\\
a_1\to a_2 & a_2\to a_3 & a_3\to a_1 & b_1\to b_2 & b_2\to b_3 & b_3\to b_1 & x_{1,2}\to x_{2,3} & x_{1,3}\to \in x_{2,1} & x_{2,1}\to x_{3,2} & x_{2,3}\to \in x_{3,1}\\
a_1\to a_3 & a_2\to a_1 & a_3\to a_2 & b_1\to b_3 & b_2\to b_1 & b_3\to b_2 & x_{1,2}\to \in x_{3,1} & x_{1,3}\to \in x_{3,2} & x_{2,1}\to \frac{x_{1,3}}{\in} & x_{2,3}\to x_{1,2}\\
a_1\to a_3 & a_2\to a_2 & a_3\to a_1 & b_1\to b_3 & b_2\to b_2 & b_3\to b_1 & x_{1,2}\to \in x_{3,2} & x_{1,3}\to \in x_{3,1} & x_{2,1}\to \frac{x_{2,3}}{\in} & x_{2,3}\to \in x_{2,1}
\end{pmatrix}$$

In[•]:=
```
n = 3;
MatrixForm@Table[u → ϕ_{1/ϵ}@Act_σ@ϕ_ϵ@u, {σ, Permutations@Range@n}, {u, Basis[n]} ]
```

Out[•]//MatrixForm=

$$\begin{pmatrix}
a_1\to a_1 & a_2\to a_2 & a_3\to a_3 & b_1\to b_1 & b_2\to b_2 & b_3\to b_3 & x_{1,2}\to x_{1,2} & x_{1,3}\to x_{1,3} & x_{2,1}\to x_{2,1} & x_{2,3}\to x_{2,3} & x\\
a_1\to a_1 & a_2\to a_3 & a_3\to a_2 & b_1\to b_1 & b_2\to b_3 & b_3\to b_2 & x_{1,2}\to x_{1,3} & x_{1,3}\to x_{1,2} & x_{2,1}\to x_{3,1} & x_{2,3}\to \frac{x_{3,2}}{\in} & x\\
a_1\to a_2 & a_2\to a_1 & a_3\to a_3 & b_1\to b_2 & b_2\to b_1 & b_3\to b_3 & x_{1,2}\to \frac{x_{2,1}}{\in} & x_{1,3}\to x_{2,3} & x_{2,1}\to \in x_{1,2} & x_{2,3}\to x_{1,3} & x\\
a_1\to a_2 & a_2\to a_3 & a_3\to a_1 & b_1\to b_2 & b_2\to b_3 & b_3\to b_1 & x_{1,2}\to x_{2,3} & x_{1,3}\to \frac{x_{2,1}}{\in} & x_{2,1}\to x_{3,2} & x_{2,3}\to \frac{x_{3,1}}{\in} & x_3\\
a_1\to a_3 & a_2\to a_1 & a_3\to a_2 & b_1\to b_3 & b_2\to b_1 & b_3\to b_2 & x_{1,2}\to \frac{x_{3,1}}{\in} & x_{1,3}\to \frac{x_{3,2}}{\in} & x_{2,1}\to \in x_{1,3} & x_{2,3}\to x_{1,2} & x_3\\
a_1\to a_3 & a_2\to a_2 & a_3\to a_1 & b_1\to b_3 & b_2\to b_2 & b_3\to b_1 & x_{1,2}\to \frac{x_{3,2}}{\in} & x_{1,3}\to \frac{x_{3,1}}{\in} & x_{2,1}\to \in x_{2,3} & x_{2,3}\to \frac{x_{2,1}}{\in} & x_3
\end{pmatrix}$$