# The Turbo-Gassner Representation

Pensieve header: The turbo Gassner representation. Continues pensieve://2016-06/.

```
In[•]:= Kδ /: Kδ_is__ := KroneckerDelta[1, Length[Union[{is}]]];
        Otherwise = True;
```

## The Burau Representation

```
In[•]:= Inverse[{{0, t}, {1, 1 - t}}] /. t → t^-1 // Simplify // Transpose // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$$

```
In[•]:= B_i_,j_[ξ_] := ξ /. v_j :↦ (1 - t) v_i + t v_j;
        B̄_i_,j_[ξ_] := ξ /. v_k_ :↦ { (1 - t^-1) v_i + t^-1 v_j   j == k
                                    { v_k                         Otherwise ;

In[•]:= {{v1, v2, v3} // B_1,3, {v1, v2, v3} // B̄_1,3}

Out[•]= {{v1, v2, (1 - t) v1 + t v3}, {v1, v2, (1 - 1/t) v1 + v3/t}}

In[•]:= {v1, v2, v3} // B_1,3 // B̄_1,3 // Expand

Out[•]= {v1, v2, v3}

In[•]:= Column@{R3l = {v1, v2, v3} // B_1,2 // B_1,3 // B_2,3,
                R3r = {v1, v2, v3} // B_2,3 // B_1,3 // B_1,2 ,
                R3l - R3r // Expand}

        {v1, (1 - t) v1 + t v2, (1 - t) v1 + t ((1 - t) v2 + t v3)}
Out[•]= {v1, (1 - t) v1 + t v2, (1 - t) ((1 - t) v1 + t v2) + t ((1 - t) v1 + t v3)}
        {0, 0, 0}
```

## The Gassner Representation

```
In[•]:= G_i_,j_[ξ_] := ξ /. v_j :↦ (1 - t_i) v_i + t_i v_j;
        Ḡ_i_,j_[ξ_] := ξ /. v_j :↦ (1 - t_i^-1) v_i + t_i^-1 v_j;

In[•]:= {v1, v2, v3} // G_1,3 // Ḡ_1,3 // Expand

Out[•]= {v1, v2, v3}

In[•]:= Column@{R3l = {v1, v2, v3} // G_1,2 // G_1,3 // G_2,3,
                R3r = {v1, v2, v3} // G_2,3 // G_1,3 // G_1,2,
                R3l - R3r // Expand}

        {v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 ((1 - t2) v2 + t2 v3)}
Out[•]= {v1, (1 - t1) v1 + t1 v2, (1 - t2) ((1 - t1) v1 + t1 v2) + t2 ((1 - t1) v1 + t1 v3)}
        {0, 0, 0}
```

```
In[•]:= Column@{OCl = {v1, v2, v3} // G1,2 // G1,3,
          OCr = {v1, v2, v3} // G1,3 // G1,2 ,
          OCl - OCr // Expand}
```

$$\{v_1,\ (1-t_1)\,v_1+t_1\,v_2,\ (1-t_1)\,v_1+t_1\,v_3\}$$

```
Out[•]=
```

$$\{v_1,\ (1-t_1)\,v_1+t_1\,v_2,\ (1-t_1)\,v_1+t_1\,v_3\}$$

$$\{0,\ 0,\ 0\}$$

```
In[•]:= Column@{UCl = {v1, v2, v3} // G1,3 // G2,3,
          UCr = {v1, v2, v3} // G2,3 // G1,3 ,
          UCl - UCr // Expand}
```

$$\{v_1,\ v_2,\ (1-t_1)\,v_1+t_1\,((1-t_2)\,v_2+t_2\,v_3)\}$$

```
Out[•]=
```

$$\{v_1,\ v_2,\ (1-t_2)\,v_2+t_2\,((1-t_1)\,v_1+t_1\,v_3)\}$$

$$\{0,\ 0,\ v_1-t_1\,v_1-t_2\,v_1+t_1\,t_2\,v_1-v_2+t_1\,v_2+t_2\,v_2-t_1\,t_2\,v_2\}$$

## The Gassner-Plus Representation

```
In[•]:= GPi_,j_[ℰ_] := Expand[ℰ /. {uj :> (1 - ti) ui + ti uj,
              f_. vj :> f (1 - ti) vi + f ti vj + (ti - 1) (ti ∂ti f - tj ∂tj f) ui + f ti ui }];
        GP‾i_,j_[ℰ_] := Expand[ℰ /. {uj :> (1 - ti^-1) ui + ti^-1 uj,
              f_. vj :> f (1 - ti^-1) vi + f ti^-1 vj + (ti^-1 - 1) (ti ∂ti f - tj ∂tj f) ui - f ti^-1 ui }];
```

```
In[•]:= GPchecks = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3};
        GPchecks // GP1,3 // GP‾1,3
```

```
Out[•]=
```

$$\{f[t_1,\ t_2,\ t_3]\,v_1,\ f[t_1,\ t_2,\ t_3]\,v_2,\ f[t_1,\ t_2,\ t_3]\,v_3,\ u_1,\ u_2,\ u_3\}$$

```
In[•]:= R3l = GPchecks // GP1,2 // GP1,3 // GP2,3
```

```
Out[•]=
```

$$\{f[t_1,\ t_2,\ t_3]\,v_1,\ f[t_1,\ t_2,\ t_3]\,t_1\,u_1+f[t_1,\ t_2,\ t_3]\,v_1-f[t_1,\ t_2,\ t_3]\,t_1\,v_1+f[t_1,\ t_2,\ t_3]\,t_1\,v_2+t_2\,u_1\,f^{(0,1,0)}[t_1,\ t_2,\ t_3]-t_1\,t_2\,u_1\,f^{(0,1,0)}[t_1,\ t_2,\ t_3]-t_1\,u_1\,f^{(1,0,0)}[t_1,\ t_2,\ t_3]+t_1^2\,u_1\,f^{(1,0,0)}[t_1,\ t_2,\ t_3],\ f[t_1,\ t_2,\ t_3]\,t_1\,u_1+f[t_1,\ t_2,\ t_3]\,t_1\,t_2\,u_2+f[t_1,\ t_2,\ t_3]\,v_1-f[t_1,\ t_2,\ t_3]\,t_1\,v_1+f[t_1,\ t_2,\ t_3]\,t_1\,v_2-f[t_1,\ t_2,\ t_3]\,t_1\,t_2\,v_2+f[t_1,\ t_2,\ t_3]\,t_1\,t_2\,v_3+t_3\,u_1\,f^{(0,0,1)}[t_1,\ t_2,\ t_3]-t_1\,t_3\,u_1\,f^{(0,0,1)}[t_1,\ t_2,\ t_3]+t_1\,t_3\,u_2\,f^{(0,0,1)}[t_1,\ t_2,\ t_3]-t_1\,t_2\,t_3\,u_2\,f^{(0,0,1)}[t_1,\ t_2,\ t_3]-t_1\,t_2\,u_2\,f^{(0,1,0)}[t_1,\ t_2,\ t_3]+t_1\,t_2^2\,u_2\,f^{(0,1,0)}[t_1,\ t_2,\ t_3]-t_1\,u_1\,f^{(1,0,0)}[t_1,\ t_2,\ t_3]+t_1^2\,u_1\,f^{(1,0,0)}[t_1,\ t_2,\ t_3],\ u_1,\ u_1-t_1\,u_1+t_1\,u_2,\ u_1-t_1\,u_1+t_1\,u_2-t_1\,t_2\,u_2+t_1\,t_2\,u_3\}$$

```
In[•]:= R3r = GPchecks // GP2,3 // GP1,3 // GP1,2; R3l - R3r
```

```
Out[•]=
```

$$\{0,\ 0,\ 0,\ 0,\ 0,\ 0\}$$

```
In[•]:= OCl = GPchecks // GP1,2 // GP1,3
```

```
Out[•]=
```

$$\{f[t_1,\ t_2,\ t_3]\,v_1,\ f[t_1,\ t_2,\ t_3]\,t_1\,u_1+f[t_1,\ t_2,\ t_3]\,v_1-f[t_1,\ t_2,\ t_3]\,t_1\,v_1+f[t_1,\ t_2,\ t_3]\,t_1\,v_2+t_2\,u_1\,f^{(0,1,0)}[t_1,\ t_2,\ t_3]-t_1\,t_2\,u_1\,f^{(0,1,0)}[t_1,\ t_2,\ t_3]-t_1\,u_1\,f^{(1,0,0)}[t_1,\ t_2,\ t_3]+t_1^2\,u_1\,f^{(1,0,0)}[t_1,\ t_2,\ t_3],\ f[t_1,\ t_2,\ t_3]\,t_1\,u_1+f[t_1,\ t_2,\ t_3]\,v_1-f[t_1,\ t_2,\ t_3]\,t_1\,v_1+f[t_1,\ t_2,\ t_3]\,t_1\,v_3+t_3\,u_1\,f^{(0,0,1)}[t_1,\ t_2,\ t_3]-t_1\,t_3\,u_1\,f^{(0,0,1)}[t_1,\ t_2,\ t_3]-t_1\,u_1\,f^{(1,0,0)}[t_1,\ t_2,\ t_3]+t_1^2\,u_1\,f^{(1,0,0)}[t_1,\ t_2,\ t_3],\ u_1,\ u_1-t_1\,u_1+t_1\,u_2,\ u_1-t_1\,u_1+t_1\,u_3\}$$

```
In[•]:= OCr = GPchecks // GP_{1,3} // GP_{1,2}; OCl - OCr
Out[•]= {0, 0, 0, 0, 0, 0}
```

Question. Does GP factor through G? How?

## The End(G) Representation

```
In[•]:= EG_{i_,j_}[ξ_] := Expand[ξ /. {u_j → (1 - t_i) u_i + t_i u_j, w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
        EGbar_{i_,j_}[ξ_] := Expand[ξ /. {u_j → (1 - t_i^-1) u_i + t_i^-1 u_j, w_i → w_i + (1 - t_i) w_j, w_j → t_i w_j}];
        EGchecks = Flatten@Table[u_i w_j, {i, 3}, {j, 3}];
        EGchecks // EG_{1,3} // EGbar_{1,3}
Out[•]= {u1 w1, u1 w2, u1 w3, u2 w1, u2 w2, u2 w3, u3 w1, u3 w2, u3 w3}
```

```
Short[R3l = EGchecks // EG_{1,2} // EG_{1,3} // EG_{2,3}, 10]
```

$$\Big\{u_1 w_1 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1},\ \frac{u_1 w_2}{t_1} + \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_1 t_2},\ \frac{u_1 w_3}{t_1 t_2},$$

$$u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 + 2 u_1 w_2 - \frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + 2 u_1 w_3 - \frac{u_1 w_3}{t_1} -$$

$$t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3,\ -u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},$$

$$-\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} + \frac{u_2 w_3}{t_2},\ u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 - t_1 t_2 u_2 w_1 + t_1 t_2 u_3 w_1 + 2 u_1 w_2 -$$

$$\frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + t_2 u_2 w_2 - t_1 t_2 u_2 w_2 - t_2 u_3 w_2 + t_1 t_2 u_3 w_2 +$$

$$2 u_1 w_3 - \frac{u_1 w_3}{t_1} - t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3 + t_2 u_2 w_3 - t_1 t_2 u_2 w_3 - t_2 u_3 w_3 + t_1 t_2 u_3 w_3,$$

$$-u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - t_2 u_2 w_2 + t_2 u_3 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + 2 u_2 w_3 -$$

$$\frac{u_2 w_3}{t_2} - t_2 u_2 w_3 - u_3 w_3 + t_2 u_3 w_3,\ -\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2} + u_3 w_3\Big\}$$

```
R3r = EGchecks // EG_{2,3} // EG_{1,3} // EG_{1,2}; R3l - R3r
{0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
(# → Collect[EG_{i,j}[#], u_ w_, Simplify]) & /@ {u_k w_j, u_k w_i, u_j w_k, u_j w_i}
```

$$\Big\{u_k w_j \to \frac{u_k w_j}{t_i},\ u_k w_i \to u_k w_i + \left(1 - \frac{1}{t_i}\right) u_k w_j,\ u_j w_k \to (1 - t_i)\, u_i w_k + t_i u_j w_k,$$

$$u_j w_i \to (1 - t_i)\, u_i w_i + t_i u_j w_i - \frac{(-1 + t_i)^2 u_i w_j}{t_i} + (-1 + t_i)\, u_j w_j\Big\}$$

## The End(G)+c Representation

Is there topology behind this representation?

In[•]:=
```
EGc_{i_,j_}[ξ_] := Expand[ξ /. {
      c_i → c_i - (1 - t_i^-1) u_i w_j, c_j → c_j + (1 - t_i^-1) u_i w_j,
      u_j → (1 - t_i) u_i + t_i u_j,
      w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
EGc̄_{i_,j_}[ξ_] := Expand[ξ /. {
      c_i → c_i + (t_i - 1) u_i w_j, c_j → c_j + (1 - t_i) u_i w_j,
      u_j → (1 - t_i^-1) u_i + t_i^-1 u_j,
      w_i → w_i + (1 - t_i) w_j, w_j → t_i w_j}];
EGcchecks = {c1, c2, c3, u1 w1, u1 w2, u1 w3, u2 w1, u2 w2, u2 w3, u3 w1, u3 w2, u3 w3};
EGcchecks // EGc_{1,3} // EGc̄_{1,3}
```

Out[•]=
$$\{c_1, c_2, c_3, u_1 w_1, u_1 w_2, u_1 w_3, u_2 w_1, u_2 w_2, u_2 w_3, u_3 w_1, u_3 w_2, u_3 w_3\}$$

```
(# → Collect[EGc_{1,2}[#], u_ w_, Simplify]) & /@ EGcchecks
```

$$\left\{c_1 \to c_1 + \left(-1 + \frac{1}{t_1}\right) u_1 w_2,\ c_2 \to c_2 + \left(1 - \frac{1}{t_1}\right) u_1 w_2,\right.$$
$$c_3 \to c_3,\ u_1 w_1 \to u_1 w_1 + \left(1 - \frac{1}{t_1}\right) u_1 w_2,\ u_1 w_2 \to \frac{u_1 w_2}{t_1},\ u_1 w_3 \to u_1 w_3,$$
$$u_2 w_1 \to \left(1 - t_1\right) u_1 w_1 + t_1 u_2 w_1 - \frac{\left(-1 + t_1\right)^2 u_1 w_2}{t_1} + \left(-1 + t_1\right) u_2 w_2,\ u_2 w_2 \to \left(-1 + \frac{1}{t_1}\right) u_1 w_2 + u_2 w_2,$$
$$\left.u_2 w_3 \to \left(1 - t_1\right) u_1 w_3 + t_1 u_2 w_3,\ u_3 w_1 \to u_3 w_1 + \left(1 - \frac{1}{t_1}\right) u_3 w_2,\ u_3 w_2 \to \frac{u_3 w_2}{t_1},\ u_3 w_3 \to u_3 w_3\right\}$$

```
u1 w2 + c2 // EGc_{1,2}
```

$$c_2 + u_1 w_2$$

```
(# → Simplify[EGc_{1,2}[#] /. {u_i_ w_i_ → 1, u_i_ w_j_ /; i ≠ j :> 0}]) & /@ EGcchecks
```

$$\{c_1 \to c_1,\ c_2 \to c_2,\ c_3 \to c_3,\ u_1 w_1 \to 1,\ u_1 w_2 \to 0,\ u_1 w_3 \to 0,$$
$$u_2 w_1 \to 0,\ u_2 w_2 \to 1,\ u_2 w_3 \to 0,\ u_3 w_1 \to 0,\ u_3 w_2 \to 0,\ u_3 w_3 \to 1\}$$

```
Short[R3l = EGcchecks // EGc1,2 // EGc1,3 // EGc2,3, 10]
```

$$\Big\{c_1 - u_1 w_2 + \frac{u_1 w_2}{t_1} - u_1 w_3 + \frac{u_1 w_3}{t_1},\ c_2 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2},$$
$$c_3 + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},\ u_1 w_1 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1},\ \frac{u_1 w_2}{t_1} + \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_1 t_2},$$
$$\frac{u_1 w_3}{t_1 t_2},\ u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 + 2 u_1 w_2 - \frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + 2 u_1 w_3 - \frac{u_1 w_3}{t_1} -$$
$$t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3,\ -u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},$$
$$-\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} + \frac{u_2 w_3}{t_2},\ u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 - t_1 t_2 u_2 w_1 + t_1 t_2 u_3 w_1 + 2 u_1 w_2 -$$
$$\frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + t_2 u_2 w_2 - t_1 t_2 u_2 w_2 - t_2 u_3 w_2 + t_1 t_2 u_3 w_2 +$$
$$2 u_1 w_3 - \frac{u_1 w_3}{t_1} - t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3 + t_2 u_2 w_3 - t_1 t_2 u_2 w_3 - t_2 u_3 w_3 + t_1 t_2 u_3 w_3,$$
$$-u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - t_2 u_2 w_2 + t_2 u_3 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + 2 u_2 w_3 -$$
$$\frac{u_2 w_3}{t_2} - t_2 u_2 w_3 - u_3 w_3 + t_2 u_3 w_3,\ -\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2} + u_3 w_3\Big\}$$

```
R3r = EGcchecks // EGc2,3 // EGc1,3 // EGc1,2; R3l - R3r
```

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

## The Turbo-Gassner Representation

```
In[•]:= TGi_,j_[ℰ_] := Expand[ℰ /. {
          f_. vk_ :> Plus[f vk /. vj → (1 - ti) vi + ti vj,
            (1 - ti^-1) (ti ∂ti f - tj ∂tj f) (uk /. uj → (1 - ti) ui + ti uj) ui wj,
            Kδk,i f (uj - ui) ui wj],
          uj → (1 - ti) ui + ti uj,
          wi → wi + (1 - ti^-1) wj, wj → ti^-1 wj}];
        TG̅i_,j_[ℰ_] := Expand[ℰ /. {
          f_. vk_ :> Plus[f vk /. vj → (1 - ti^-1) vi + ti^-1 vj,
            (1 - ti) (ti ∂ti f - tj ∂tj f) (uk /. uj → (1 - ti^-1) ui + ti^-1 uj) ui wj,
            Kδk,i f (ui - uj) ui wj],
          uj → (1 - ti^-1) ui + ti^-1 uj,
          wi → wi + (1 - ti) wj, wj → ti wj}];
        TGchecks = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3, w1, w2, w3};
        TGchecks // TG1,3 // TG̅1,3
Out[•]= {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3, w1, w2, w3}
```

```
In[•]:= Short[R3l = TGchecks // TG1,2 // TG1,3 // TG2,3, 10]
```

Out[•]//Short=

$$\Big\{f[t_1,t_2,t_3]\,v_1 - f[t_1,t_2,t_3]\,u_1^2\,w_2 + f[t_1,t_2,t_3]\,u_1\,u_2\,w_2 - f[t_1,t_2,t_3]\,u_1^2\,w_3 + f[t_1,t_2,t_3]\,u_1\,u_3\,w_3 - \frac{t_3\,u_1^2\,w_3\,f^{(0,0,1)}[t_1,t_2,t_3]}{t_2} + \text{<<15>>} + t_2\,u_1\,u_2\,w_3\,f^{(0,1,0)}[t_1,t_2,t_3] - u_1^2\,w_2\,f^{(1,0,0)}[t_1,t_2,t_3] + t_1\,u_1^2\,w_2\,f^{(1,0,0)}[t_1,t_2,t_3] - u_1^2\,w_3\,f^{(1,0,0)}[t_1,t_2,t_3] + t_1\,u_1^2\,w_3\,f^{(1,0,0)}[t_1,t_2,t_3],\ f[t_1,t_2,t_3]\,v_1 - \text{<<1>>}\,t_1\,v_1 + \text{<<63>>} + t_1^2\,u_1\,u_2\,w_3\,f^{(1,0,0)}[t_1,t_2,t_3],\ \text{<<5>>},\ \text{<<1>>},\ \frac{w_3}{t_1\,t_2}\Big\}$$

```
In[•]:= R3r = TGchecks // TG2,3 // TG1,3 // TG1,2; R3l - R3r
```

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0}

```
In[•]:= Short[OCl = TGchecks // TG1,2 // TG1,3]
```

Out[•]//Short=

$$\Big\{\text{<<18>>} + t_1\,\text{<<2>>}\,f^{\text{<<1>>}}[t_1,t_2,t_3],\ \text{<<1>>},\ \text{<<6>>},\ \frac{\text{<<1>>}}{\text{<<1>>}}\Big\}$$

```
In[•]:= OCr = TGchecks // TG1,3 // TG1,2; OCl - OCr
```

Out[•]= {0, −f[t1, t2, t3] u1 u2 w3 + f[t1, t2, t3] t1 u1 u2 w3 + f[t1, t2, t3] u1 u3 w3 − f[t1, t2, t3] t1 u1 u3 w3, −f[t1, t2, t3] u1 u2 w2 + f[t1, t2, t3] t1 u1 u2 w2 + f[t1, t2, t3] u1 u3 w2 − f[t1, t2, t3] t1 u1 u3 w2, 0, 0, 0, 0, 0, 0}

## A Finite-Rank Turbo-Gassner Representation

```
In[•]:= FTG_i_,j_[ξ_] := Expand[ξ /. {
          v_j → (1 - t_i) v_i + t_i v_j - v_i,i + v_i,j,
          v_i → v_i + (u_j - u_i) u_i w_j,
          v_l_,k_ :→ v_l,k + Kδ_k,j (1 - t_i) (v_l,i - v_l,j) +
             (1 - t_i^-1) (t_i Kδ_l,i - t_j Kδ_l,j) (u_k /. u_j → (1 - t_i) u_i + t_i u_j) u_i w_j,
          u_j → (1 - t_i) u_i + t_i u_j,
          w_i → w_i + (1 - t_i^-1) w_j,  w_j → t_i^-1 w_j}];
        FTG‾_i_,j_[ξ_] := Expand[ξ /. {
          v_j → (1 - t_i^-1) v_i + t_i^-1 v_j + t_i^-2 v_i,i - t_i^-2 v_i,j,
          v_i → v_i + (u_i - u_j) u_i w_j,
          v_l_,k_ :→ v_l,k + Kδ_k,j (1 - t_i^-1) (v_l,i - v_l,j) +
             (1 - t_i) (t_i Kδ_l,i - t_j Kδ_l,j) (u_k /. u_j → (1 - t_i^-1) u_i + t_i^-1 u_j) u_i w_j,
          u_j → (1 - t_i^-1) u_i + t_i^-1 u_j,
          w_i → w_i + (1 - t_i) w_j,  w_j → t_i w_j}];
        ff = f0 + f1 η[1] + f2 η[2] + f3 η[3];
        FTGchecks[n_] :=
          {Table[v_k, {k, n}], Table[v_1,k, {l, n}, {k, n}], Table[u_k, {k, n}], Table[w_k, {k, n}]};
        (FTGchecks[3] // TG1,3 // FTG‾1,3) - FTGchecks[3]
```

Out[•]= {{0, 0, 0}, {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}, {0, 0, 0}, {0, 0, 0}}

In[•]:=
```
FTGchecks[3] // FTG1,2 // Column
```

Out[•]=
$$\left\{v_1 - u_1^2 w_2 + u_1 u_2 w_2,\ v_1 - t_1 v_1 + t_1 v_2 - v_{1,1} + v_{1,2},\ v_3\right\}$$
$$\left\{\left\{-u_1^2 w_2 + t_1 u_1^2 w_2 + v_{1,1},\ -u_1^2 w_2 + 2 t_1 u_1^2 w_2 - t_1^2 u_1^2 w_2 - t_1 u_1 u_2 w_2 + t_1^2 u_1 u_2 w_2 + v_{1,1} - t_1 v_{1,1} + t_1 v_{1,2},\right.\right.$$
$$\left.-u_1 u_3 w_2 + t_1 u_1 u_3 w_2 + v_{1,3}\right\},\ \left\{-t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + v_{2,1},\right.$$
$$\left.-2 t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + t_1 t_2 u_1^2 w_2 + t_2 u_1 u_2 w_2 - t_1 t_2 u_1 u_2 w_2 + v_{2,1} - t_1 v_{2,1} + t_1 v_{2,2},\right.$$
$$\left.\left.-t_2 u_1 u_3 w_2 + \frac{t_2 u_1 u_3 w_2}{t_1} + v_{2,3}\right\},\ \left\{v_{3,1},\ v_{3,1} - t_1 v_{3,1} + t_1 v_{3,2},\ v_{3,3}\right\}\right\}$$
$$\left\{u_1,\ u_1 - t_1 u_1 + t_1 u_2,\ u_3\right\}$$
$$\left\{w_1 + w_2 - \frac{w_2}{t_1},\ \frac{w_2}{t_1},\ w_3\right\}$$

In[•]:=
```
Short[R3l = FTGchecks[3] // FTG1,2 // FTG1,3 // FTG2,3, 10]
```

Out[•]//Short=
$$\left\{\left\{v_1 - u_1^2 w_2 + u_1 u_2 w_2 - u_1^2 w_3 + u_1 u_3 w_3,\right.\right.$$
$$v_1 - t_1 v_1 + t_1 v_2 - u_1 u_2 w_3 + t_1 u_1 u_2 w_3 - t_1 u_2^2 w_3 + u_1 u_3 w_3 - t_1 u_1 u_3 w_3 + t_1 u_2 u_3 w_3 - v_{1,1} + v_{1,2},$$
$$\left.v_1 - t_1 v_1 + t_1 v_2 - t_1 t_2 v_2 + t_1 t_2 v_3 - v_{1,1} + v_{1,2} - t_2 v_{1,2} + t_2 v_{1,3} - t_1 v_{2,2} + t_1 v_{2,3}\right\},$$
$$\text{<<1>>},\ \left\{u_1,\ u_1 - t_1 u_1 + t_1 u_2,\ u_1 - t_1 u_1 + t_1 u_2 - t_1 t_2 u_2 + t_1 t_2 u_3\right\},$$
$$\left.\left\{w_1 + w_2 - \frac{w_2}{t_1} + w_3 - \frac{w_3}{t_1},\ \frac{w_2}{t_1} + \frac{w_3}{t_1} - \frac{w_3}{t_1 t_2},\ \frac{w_3}{t_1 t_2}\right\}\right\}$$

In[•]:=
```
R3r = FTGchecks[3] // FTG2,3 // FTG1,3 // FTG1,2; R3l - R3r
```

Out[•]=
$$\{\{0, 0, 0\}, \{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\}, \{0, 0, 0\}, \{0, 0, 0\}\}$$

In[•]:=
```
Short[OCl = FTGchecks[3] // FTG1,2 // FTG1,3, 100]
```

Out[•]//Short=
$$\left\{\left\{v_1 - u_1^2 w_2 + u_1 u_2 w_2 - u_1^2 w_3 + u_1 u_3 w_3,\ v_1 - t_1 v_1 + t_1 v_2 - u_1 u_2 w_3 +\right.\right.$$
$$t_1 u_1 u_2 w_3 + u_1 u_3 w_3 - t_1 u_1 u_3 w_3 - v_{1,1} + v_{1,2},\ v_1 - t_1 v_1 + t_1 v_3 - v_{1,1} + v_{1,3}\},$$
$$\left\{\left\{-u_1^2 w_2 + t_1 u_1^2 w_2 - u_1^2 w_3 + t_1 u_1^2 w_3 + v_{1,1},\ -u_1^2 w_2 + 2 t_1 u_1^2 w_2 - t_1^2 u_1^2 w_2 - t_1 u_1 u_2 w_2 +\right.\right.$$
$$t_1^2 u_1 u_2 w_2 - u_1^2 w_3 + 2 t_1 u_1^2 w_3 - t_1^2 u_1^2 w_3 - t_1 u_1 u_2 w_3 + t_1^2 u_1 u_2 w_3 + v_{1,1} - t_1 v_{1,1} + t_1 v_{1,2},$$
$$-u_1^2 w_2 + 2 t_1 u_1^2 w_2 - t_1^2 u_1^2 w_2 - t_1 u_1 u_3 w_2 + t_1^2 u_1 u_3 w_2 - u_1^2 w_3 + 2 t_1 u_1^2 w_3 - t_1^2 u_1^2 w_3 -$$
$$\left.t_1 u_1 u_3 w_3 + t_1^2 u_1 u_3 w_3 + v_{1,1} - t_1 v_{1,1} + t_1 v_{1,3}\right\},\ \left\{-t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + v_{2,1},\right.$$
$$-2 t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + t_1 t_2 u_1^2 w_2 + t_2 u_1 u_2 w_2 - t_1 t_2 u_1 u_2 w_2 + v_{2,1} - t_1 v_{2,1} + t_1 v_{2,2},$$
$$\left.-2 t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + t_1 t_2 u_1^2 w_2 + t_2 u_1 u_3 w_2 - t_1 t_2 u_1 u_3 w_2 + v_{2,1} - t_1 v_{2,1} + t_1 v_{2,3}\right\},$$
$$\left\{-t_3 u_1^2 w_3 + \frac{t_3 u_1^2 w_3}{t_1} + v_{3,1},\ -2 t_3 u_1^2 w_3 + \frac{t_3 u_1^2 w_3}{t_1} + t_1 t_3 u_1^2 w_3 +\right.$$
$$t_3 u_1 u_2 w_3 - t_1 t_3 u_1 u_2 w_3 + v_{3,1} - t_1 v_{3,1} + t_1 v_{3,2},$$
$$\left.\left.-2 t_3 u_1^2 w_3 + \frac{t_3 u_1^2 w_3}{t_1} + t_1 t_3 u_1^2 w_3 + t_3 u_1 u_3 w_3 - t_1 t_3 u_1 u_3 w_3 + v_{3,1} - t_1 v_{3,1} + t_1 v_{3,3}\right\}\right\},$$
$$\left.\left\{u_1,\ u_1 - t_1 u_1 + t_1 u_2,\ u_1 - t_1 u_1 + t_1 u_3\right\},\ \left\{w_1 + w_2 - \frac{w_2}{t_1} + w_3 - \frac{w_3}{t_1},\ \frac{w_2}{t_1},\ \frac{w_3}{t_1}\right\}\right\}$$

In[•]:= **Short[OCr = FTGchecks[3] // FTG$_{1,3}$ // FTG$_{1,2}$, 100]**

Out[•]//Short=
$$\Big\{\Big\{v_1 - u_1^2 w_2 + u_1 u_2 w_2 - u_1^2 w_3 + u_1 u_3 w_3,\ v_1 - t_1 v_1 + t_1 v_2 - v_{1,1} + v_{1,2},$$
$$v_1 - t_1 v_1 + t_1 v_3 + u_1 u_2 w_2 - t_1 u_1 u_2 w_2 - u_1 u_3 w_2 + t_1 u_1 u_3 w_2 - v_{1,1} + v_{1,3}\Big\},$$
$$\Big\{\Big\{-u_1^2 w_2 + t_1 u_1^2 w_2 - u_1^2 w_3 + t_1 u_1^2 w_3 + v_{1,1},\ -u_1^2 w_2 + 2 t_1 u_1^2 w_2 - t_1^2 u_1^2 w_2 - t_1 u_1 u_2 w_2 +$$
$$t_1^2 u_1 u_2 w_2 - u_1^2 w_3 + 2 t_1 u_1^2 w_3 - t_1^2 u_1^2 w_3 - t_1 u_1 u_2 w_3 + t_1^2 u_1 u_2 w_3 + v_{1,1} - t_1 v_{1,1} + t_1 v_{1,2},$$
$$-u_1^2 w_2 + 2 t_1 u_1^2 w_2 - t_1^2 u_1^2 w_2 - t_1 u_1 u_3 w_2 + t_1^2 u_1 u_3 w_2 - u_1^2 w_3 + 2 t_1 u_1^2 w_3 - t_1^2 u_1^2 w_3 -$$
$$t_1 u_1 u_3 w_3 + t_1^2 u_1 u_3 w_3 + v_{1,1} - t_1 v_{1,1} + t_1 v_{1,3}\Big\},\ \Big\{-t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + v_{2,1},$$
$$-2 t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + t_1 t_2 u_1^2 w_2 + t_2 u_1 u_2 w_2 - t_1 t_2 u_1 u_2 w_2 + v_{2,1} - t_1 v_{2,1} + t_1 v_{2,2},$$
$$-2 t_2 u_1^2 w_2 + \frac{t_2 u_1^2 w_2}{t_1} + t_1 t_2 u_1^2 w_2 + t_2 u_1 u_3 w_2 - t_1 t_2 u_1 u_3 w_2 + v_{2,1} - t_1 v_{2,1} + t_1 v_{2,3}\Big\},$$
$$\Big\{-t_3 u_1^2 w_3 + \frac{t_3 u_1^2 w_3}{t_1} + v_{3,1},\ -2 t_3 u_1^2 w_3 + \frac{t_3 u_1^2 w_3}{t_1} + t_1 t_3 u_1^2 w_3 +$$
$$t_3 u_1 u_2 w_3 - t_1 t_3 u_1 u_2 w_3 + v_{3,1} - t_1 v_{3,1} + t_1 v_{3,2},$$
$$-2 t_3 u_1^2 w_3 + \frac{t_3 u_1^2 w_3}{t_1} + t_1 t_3 u_1^2 w_3 + t_3 u_1 u_3 w_3 - t_1 t_3 u_1 u_3 w_3 + v_{3,1} - t_1 v_{3,1} + t_1 v_{3,3}\Big\}\Big\},$$
$$\{u_1,\ u_1 - t_1 u_1 + t_1 u_2,\ u_1 - t_1 u_1 + t_1 u_3\},\ \Big\{w_1 + w_2 - \frac{w_2}{t_1} + w_3 - \frac{w_3}{t_1},\ \frac{w_2}{t_1},\ \frac{w_3}{t_1}\Big\}\Big\}$$

In[•]:= **OCl - OCr**

Out[•]= $\{\{0,\ -u_1 u_2 w_3 + t_1 u_1 u_2 w_3 + u_1 u_3 w_3 - t_1 u_1 u_3 w_3,\ -u_1 u_2 w_2 + t_1 u_1 u_2 w_2 + u_1 u_3 w_2 - t_1 u_1 u_3 w_2\},$
$\{\{0, 0, 0\}, \{0, 0, 0\}, \{0, 0, 0\}\}, \{0, 0, 0\}, \{0, 0, 0\}\}$

In[•]:= **Short[FOC1 = FTGchecks[4] // FTG$_{1,2}$ // FTG$_{1,3}$ // FTG$_{1,4}$]**

Out[•]//Short= { <<1>> }

In[•]:= **Short[FOC2 = FTGchecks[4] // FTG$_{1,3}$ // FTG$_{1,2}$ // FTG$_{1,4}$]**

Out[•]//Short= { <<1>> }

In[•]:= **Short[FOC3 = FTGchecks[4] // FTG$_{1,4}$ // FTG$_{1,2}$ // FTG$_{1,3}$]**

Out[•]//Short= { <<1>> }

In[•]:= **Short[FOC4 = FTGchecks[4] // FTG$_{1,4}$ // FTG$_{1,3}$ // FTG$_{1,2}$]**

Out[•]//Short= { <<1>> }

In[•]:= **FOC1 - FOC2 - FOC3 + FOC4**

Out[•]= $\{\{0, 0, 0, 0\}, \{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\},$
$\{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}$

In[•]:= **FOC1 - FOC2**

Out[•]= $\{\{0,\ -u_1 u_2 w_3 + t_1 u_1 u_2 w_3 + u_1 u_3 w_3 - t_1 u_1 u_3 w_3,\ -u_1 u_2 w_2 + t_1 u_1 u_2 w_2 + u_1 u_3 w_2 - t_1 u_1 u_3 w_2,\ 0\},$
$\{\{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}, \{0, 0, 0, 0\}, \{0, 0, 0, 0\}\}$

In[•]:= **FOC2 - FOC3**

Out[•]= $\{\{0, u_1 u_2 w_3 - t_1 u_1 u_2 w_3 - u_1 u_3 w_3 + t_1 u_1 u_3 w_3 - u_1 u_2 w_4 + t_1 u_1 u_2 w_4 + u_1 u_4 w_4 - t_1 u_1 u_4 w_4,$
$u_1 u_2 w_2 - t_1 u_1 u_2 w_2 - u_1 u_3 w_2 + t_1 u_1 u_3 w_2 - u_1 u_3 w_4 + t_1 u_1 u_3 w_4 + u_1 u_4 w_4 - t_1 u_1 u_4 w_4,$
$-u_1 u_2 w_2 + t_1 u_1 u_2 w_2 + u_1 u_4 w_2 - t_1 u_1 u_4 w_2 - u_1 u_3 w_3 + t_1 u_1 u_3 w_3 + u_1 u_4 w_3 - t_1 u_1 u_4 w_3\},$
$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{0,0,0,0\},\{0,0,0,0\}\}$

In[•]:= **FOC1 - FOC3**

Out[•]= $\{\{0, -u_1 u_2 w_4 + t_1 u_1 u_2 w_4 + u_1 u_4 w_4 - t_1 u_1 u_4 w_4, -u_1 u_3 w_4 + t_1 u_1 u_3 w_4 + u_1 u_4 w_4 - t_1 u_1 u_4 w_4,$
$-u_1 u_2 w_2 + t_1 u_1 u_2 w_2 + u_1 u_4 w_2 - t_1 u_1 u_4 w_2 - u_1 u_3 w_3 + t_1 u_1 u_3 w_3 + u_1 u_4 w_3 - t_1 u_1 u_4 w_3\},$
$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{0,0,0,0\},\{0,0,0,0\}\}$

In[•]:= **(FTGchecks[4] // $FTG_{1,2}$ // $FTG_{1,3}$ // $\overline{FTG}_{1,2}$ // $\overline{FTG}_{1,3}$ // $FTG_{1,4}$ // $FTG_{1,3}$ // $FTG_{1,2}$ // $\overline{FTG}_{1,3}$ // $\overline{FTG}_{1,2}$ // $\overline{FTG}_{1,4}$) - FTGchecks[4]**

Out[•]= $\{\{0,0,0,0\},\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},$
$\{0,0,0,0\},\{0,0,0,0\}\}$

In[•]:= **(FTGchecks[4] // $FTG_{1,2}$ // $FTG_{1,3}$ // $\overline{FTG}_{1,2}$ // $\overline{FTG}_{1,3}$ // $FTG_{1,3}$ // $FTG_{1,3}$ // $FTG_{1,2}$ // $\overline{FTG}_{1,3}$ // $\overline{FTG}_{1,2}$ // $\overline{FTG}_{1,3}$) - FTGchecks[4]**

Out[•]= $\{\{0, u_1^2 w_3 - 2 t_1 u_1^2 w_3 + t_1^2 u_1^2 w_3 - u_1 u_2 w_3 + 2 t_1 u_1 u_2 w_3 - t_1^2 u_1 u_2 w_3,$
$u_1^2 w_2 - 2 t_1 u_1^2 w_2 + t_1^2 u_1^2 w_2 - u_1 u_2 w_2 + 2 t_1 u_1 u_2 w_2 - t_1^2 u_1 u_2 w_2, 0\},$
$\{\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\},\{0,0,0,0\}\},\{0,0,0,0\},\{0,0,0,0\}\}$

In[•]:= **(FTGchecks[5] // $FTG_{1,2}$ // $FTG_{1,3}$ // $\overline{FTG}_{1,2}$ // $\overline{FTG}_{1,3}$ // $FTG_{1,4}$ // $FTG_{1,5}$ // $\overline{FTG}_{1,4}$ // $\overline{FTG}_{1,5}$) - FTGchecks[5]**

Out[•]= $\{\{0, -u_1 u_2 w_3 + t_1 u_1 u_2 w_3 + u_1 u_3 w_3 - t_1 u_1 u_3 w_3, -u_1 u_2 w_2 + t_1 u_1 u_2 w_2 + u_1 u_3 w_2 - t_1 u_1 u_3 w_2,$
$-u_1 u_4 w_5 + t_1 u_1 u_4 w_5 + u_1 u_5 w_5 - t_1 u_1 u_5 w_5, -u_1 u_4 w_4 + t_1 u_1 u_4 w_4 + u_1 u_5 w_4 - t_1 u_1 u_5 w_4\},$
$\{\{0,0,0,0,0\},\{0,0,0,0,0\},\{0,0,0,0,0\},\{0,0,0,0,0\},\{0,0,0,0,0\}\},$
$\{0,0,0,0,0\},\{0,0,0,0,0\}\}$

## The Turbo-Burau Representation

In[•]:=
```
η /: η[i_]^2 = 0; η /: η[i_] η[j_] = 0;
TB_{i_,j_}[ℰ_] :=
  Expand[ℰ /. {
    f_. v_k_ :> Plus[f v_k /. v_j → (1 - t - η[i]) v_i + (t + η[i]) v_j,
      (t - 1) (∂_η[i] f - ∂_η[j] f) (u_k /. u_j → (1 - t) u_i + t u_j) u_i w_j,
      Kδ_{k,i} (f /. _η → 0) (u_j - u_i) u_i w_j],
    u_j → (1 - t) u_i + t u_j,
    w_i → w_i + (1 - t^-1) w_j, w_j → t^-1 w_j}];
ff = f_0 + f_1 η[1] + f_2 η[2] + f_3 η[3];
TBchecks = {ff v_1, ff v_2, ff v_3, u_1^2 w_1, u_1^2 w_2, u_1, u_2, u_3, w_1, w_2, w_3};
```

```
In[•]:= Short[R3l = TBchecks // TB1,2 // TB1,3 // TB2,3, 10]
```

Out[•]//Short=
$$\{f_0\,v_1 - f_0\,u_1^2\,w_2 - f_1\,u_1^2\,w_2 + t\,f_1\,u_1^2\,w_2 + f_2\,u_1^2\,w_2 - t\,f_2\,u_1^2\,w_2 + f_0\,u_1\,u_2\,w_2 - f_0\,u_1^2\,w_3 - f_1\,u_1^2\,w_3 + t\,f_1\,u_1^2\,w_3 + 2\,f_2\,u_1^2\,w_3 - \frac{f_2\,u_1^2\,w_3}{t} - t\,f_2\,u_1^2\,w_3 - f_3\,u_1^2\,w_3 + \frac{f_3\,u_1^2\,w_3}{t} - f_2\,u_1\,u_2\,w_3 + t\,f_2\,u_1\,u_2\,w_3 + f_3\,u_1\,u_2\,w_3 - t\,f_3\,u_1\,u_2\,w_3 + f_0\,u_1\,u_3\,w_3 + f_1\,v_1\,\eta[1] + f_2\,v_1\,\eta[2] + f_3\,v_1\,\eta[3],\ \text{<<9>>},\ \frac{w_3}{t^2}\}$$

```
In[•]:= R3r = TBchecks // TB2,3 // TB1,3 // TB1,2; R3l - R3r
```

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

```
In[•]:= Short[OCl = TBchecks // TB1,2 // TB1,3]
```

Out[•]//Short=
$$\{f_0\,v_1 - f_0\,u_1^2\,w_2 - f_1\,u_1^2\,w_2 + t\,f_1\,u_1^2\,w_2 + \text{<<12>>} + f_0\,u_1\,u_3\,w_3 + f_1\,v_1\,\eta[1] + f_2\,v_1\,\eta[2] + f_3\,v_1\,\eta[3],\ \text{<<1>>},\ \text{<<1>>},\ \text{<<5>>},\ w_1 + \text{<<6>>},\ \frac{w_2}{t},\ \frac{w_3}{t}\}$$

```
In[•]:= OCr = TBchecks // TB1,3 // TB1,2; OCl - OCr
```

Out[•]=
$$\{0,\ -f_0\,u_1\,u_2\,w_3 + t\,f_0\,u_1\,u_2\,w_3 + f_0\,u_1\,u_3\,w_3 - t\,f_0\,u_1\,u_3\,w_3,\ -f_0\,u_1\,u_2\,w_2 + t\,f_0\,u_1\,u_2\,w_2 + f_0\,u_1\,u_3\,w_2 - t\,f_0\,u_1\,u_3\,w_2, 0, 0, 0, 0, 0, 0, 0, 0\}$$

## The Turbo@1 Burau (Non) Representation

```
In[•]:= TB1i_,j_[ξ_] :=
          Expand[ξ /. {
              vk_ :> (vk /. vj → (1 - t) vi + t vj + Kδ1,i (v1j - v1i)) + Kδk,i (uj - ui) ui wj,
              v1k_ :>
                (v1k /. v1j → (1 - t) v1i + t v1j) + (t - 1) (Kδ1,i - Kδ1,j) (uk /. uj → (1 - t) ui + t uj) ui wj,
              uj → (1 - t) ui + t uj,
              wi → wi + (1 - t^-1) wj, wj → t^-1 wj}];
        TB1checks = {v1, v2, v3, v11, v12, v13, u1^2 w1, u1^2 w2, u1, u2, u3, w1, w2, w3};

In[•]:= Short[R3l = TB1checks // TB11,2 // TB11,3 // TB12,3, 10]
```

Out[•]//Short=
$$\{v_1 - u_1^2\,w_2 + u_1\,u_2\,w_2 - u_1^2\,w_3 + u_1\,u_3\,w_3,\ v_1 - t\,v_1 + t\,v_2 - v1_1 + v1_2 - u_1\,u_2\,w_3 + t\,u_1\,u_2\,w_3 - t\,u_2^2\,w_3 + u_1\,u_3\,w_3 - t\,u_1\,u_3\,w_3 + t\,u_2\,u_3\,w_3,\ v_1 - t\,v_1 + t\,v_2 - t^2\,v_2 + t^2\,v_3 - v1_1 + v1_2 - t\,v1_2 + t\,v1_3,\ v1_1 - u_1^2\,w_2 + t\,u_1^2\,w_2 - u_1^2\,w_3 + t\,u_1^2\,w_3,\ v1_1 - t\,v1_1 + t\,v1_2 - u_1^2\,w_2 + 2\,t\,u_1^2\,w_2 - t^2\,u_1^2\,w_2 - t\,u_1\,u_2\,w_2 + t^2\,u_1\,u_2\,w_2 - u_1^2\,w_3 + 2\,t\,u_1^2\,w_3 - t^2\,u_1^2\,w_3 - t\,u_1\,u_2\,w_3 + t^2\,u_1\,u_2\,w_3,\ v1_1 - t\,v1_1 + t\,v1_2 - t^2\,v1_2 + t^2\,v1_3 - u_1^2\,w_2 + 2\,t\,u_1^2\,w_2 - t^2\,u_1^2\,w_2 - t\,u_1\,u_2\,w_2 + 2\,t^2\,u_1\,u_2\,w_2 - t^3\,u_1\,u_2\,w_2 - t^2\,u_1\,u_3\,w_2 + t^3\,u_1\,u_3\,w_2 - u_1^2\,w_3 + 2\,t\,u_1^2\,w_3 - t^2\,u_1^2\,w_3 - t\,u_1\,u_2\,w_3 + 2\,t^2\,u_1\,u_2\,w_3 - t^3\,u_1\,u_2\,w_3 - t^2\,u_1\,u_3\,w_3 + t^3\,u_1\,u_3\,w_3,\ u_1^2\,w_1 + u_1^2\,w_2 - \frac{u_1^2\,w_2}{t} + u_1^2\,w_3 - \frac{u_1^2\,w_3}{t},\ \frac{u_1^2\,w_2}{t} - \frac{u_1^2\,w_3}{t^2} + \frac{u_1^2\,w_3}{t},\ u_1,\ u_1 - t\,u_1 + t\,u_2,\ u_1 - t\,u_1 + t\,u_2 - t^2\,u_2 + t^2\,u_3,\ w_1 + w_2 - \frac{w_2}{t} + w_3 - \frac{w_3}{t},\ \frac{w_2}{t} - \frac{w_3}{t^2} + \frac{w_3}{t},\ \frac{w_3}{t^2}\}$$

```
In[•]:= R3r = TB1checks // TB12,3 // TB11,3 // TB11,2; R3l - R3r
```

Out[•]=
$$\{0, 0, -t\,u_1\,u_2\,w_2 + t^2\,u_1\,u_2\,w_2 + t\,u_1\,u_3\,w_2 - t^2\,u_1\,u_3\,w_2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$