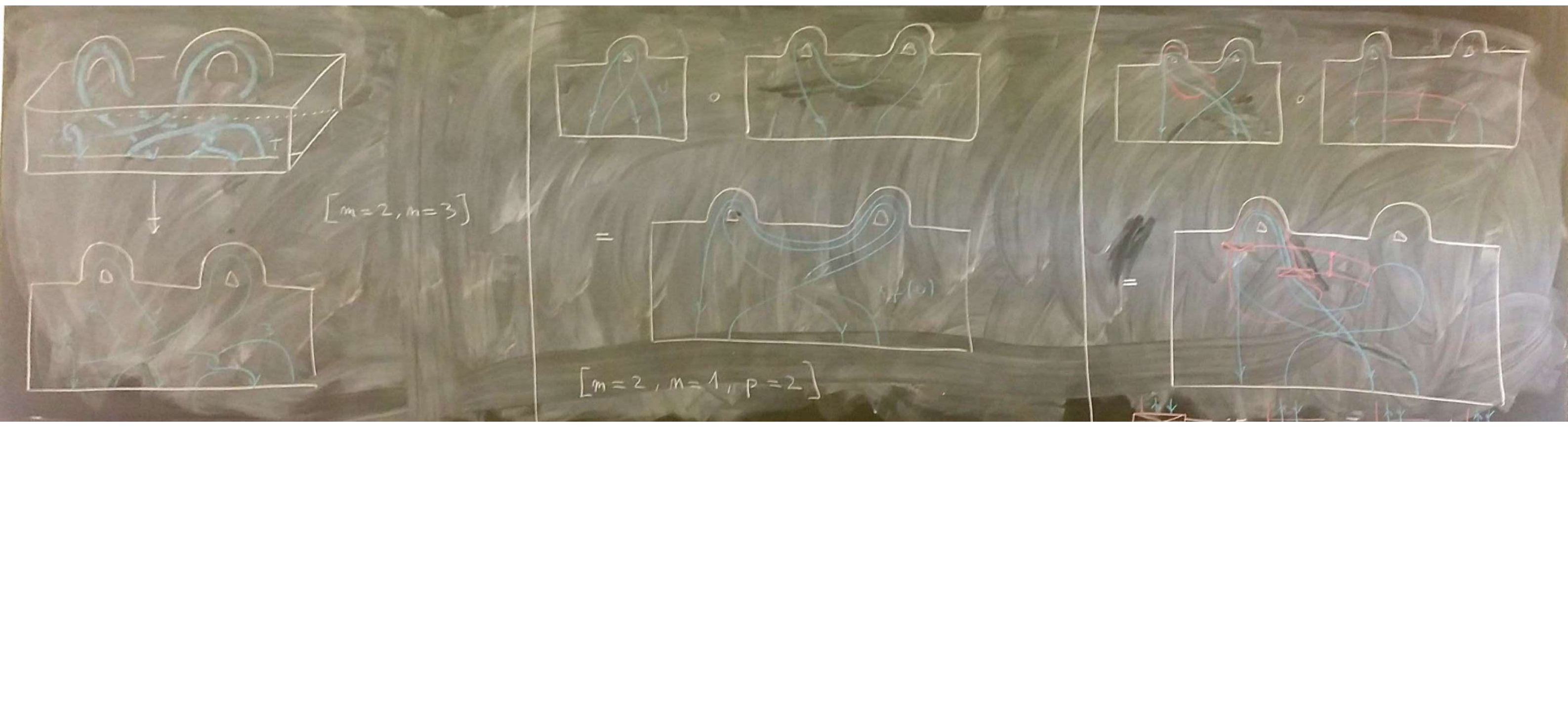


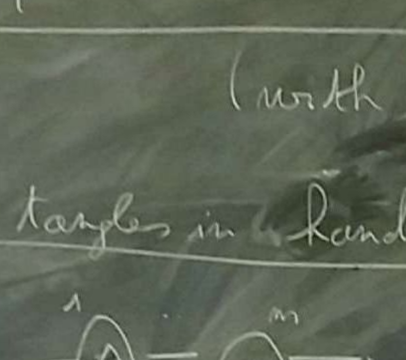
[m = 2, n = 3]
=
[m = 2, n = 1, p = 2]
=

The Kontsevich integral for bottom tangles in handlebodies: construction & properties

(with K. HABIRO)

I) The cat of bottom tangles in handlebodies

$\forall m \geq 0$, $V_m :=$



$\subset \mathbb{R}^3$

$V_m \xrightarrow{T} V_m$ rel S

An n-component bottom tangle in V_m is $(\cap_1 \cdots \cap_n) \times [0,1] \xhookrightarrow{T} V_m$ whose endpoints are "neatly aligned"

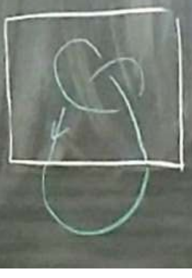
Def (Habiro '06)

Let $\mathcal{B}$ be the category where

- objects are $m \geq 0$.
- morphisms $m \to n$ are n-component bottom tangles in V_m
- $\underset{\mathcal{B}(n,p)}{\underset{\cap}{U}} \circ \underset{\mathcal{B}(m,n)}{\underset{\cap}{T}} = \underset{\mathcal{B}(m,p)}{\underset{\cap}{i_T(U)}}$

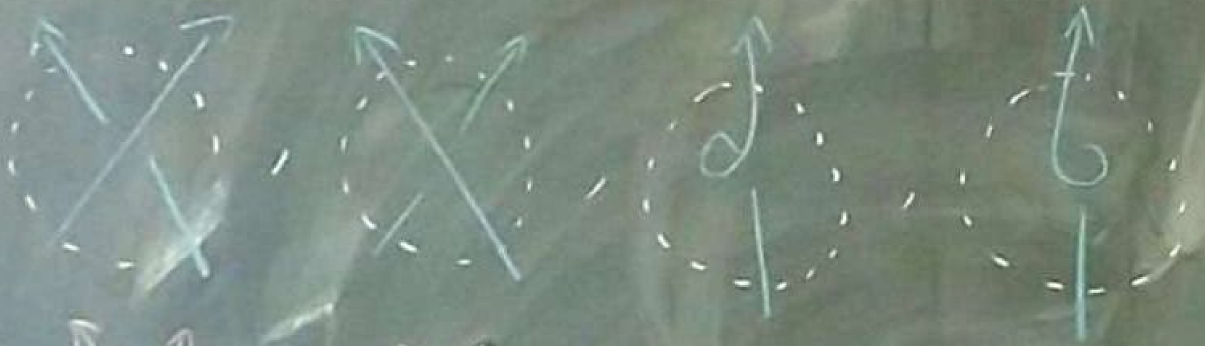
$i_{U \circ T} = i_T \circ i_U$

ex $\quad id_m =$ 

eg $\mathcal{B}(0,1) \cong \{\text{knots in } S^3\}$ 

$$\begin{array}{ccc} \mathcal{B}(m,m)^\times & \cong & MCG(V_m, S)^{op} \\ \cup & & \\ T & \longmapsto & i_T \end{array}$$

Let $T \in B(m,n)$, fix a diagram of T and let D be the choice of k disks in that diagram where

$T =$ 

$T_P =$ 

$$[T,D] = \sum_{P \subset D} (-1)^{\#P} \cdot T_P \in \mathbb{Q} \cdot B(m,n)$$

$$\mathcal{V}^k(m,n) = \langle [T;D] \mid T \in \mathcal{B}(m,n),\ \#D = k \rangle \leq \mathbb{Q}\mathcal{B}(m,n)$$

Vassiliev filtration

$$\mathbb{Q}\cdot\mathcal{B} = \mathcal{V}^0 \supset \mathcal{V}^1 \supset \cdots \mathcal{V}^k \supset \mathcal{V}^{k+1} \supset \cdots$$

$$\mathbb{Q}\mathcal{B}/\mathcal{V}^1 \longleftarrow \mathbb{Q}\mathcal{B}/\mathcal{V}^2 \longleftarrow \cdots \longleftarrow \mathbb{Q}\mathcal{B}/\mathcal{V}^k \longleftarrow \cdots$$

$\cong \mathbb{Q}\cdot\mathcal{F}^{op}$ where $\mathcal{F}$ is the cat of finitely gen free groups

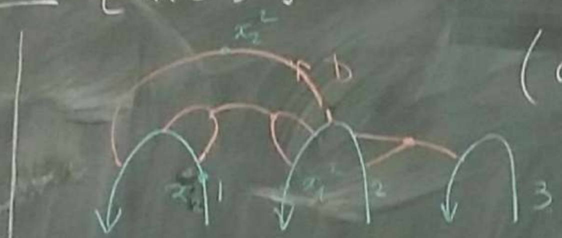
TODAY'S GOAL

Describe the associated graded $\mathrm{Gr}^{\mathcal{V}} QB = \bigoplus_{k=0}^{\infty} \mathcal{V}^k/\mathcal{V}^{k+1}$

(II) The cat of Jacobi diagrams in handlebodies

A Jacobi diagram D on $\cap^1 \cdots \cap^n$ is a unitrivalent graph whose trivalent vert are oriented and whose set of univalent vert is embedded in $\cap^1 \cdots \cap^n$

Ex $[n=3]$



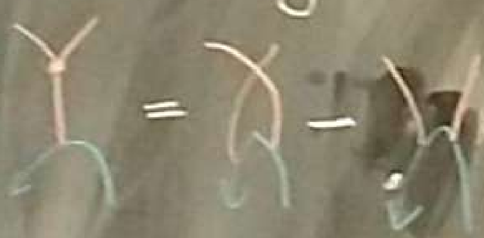
$(\deg D = 6)$ $\rightsquigarrow$

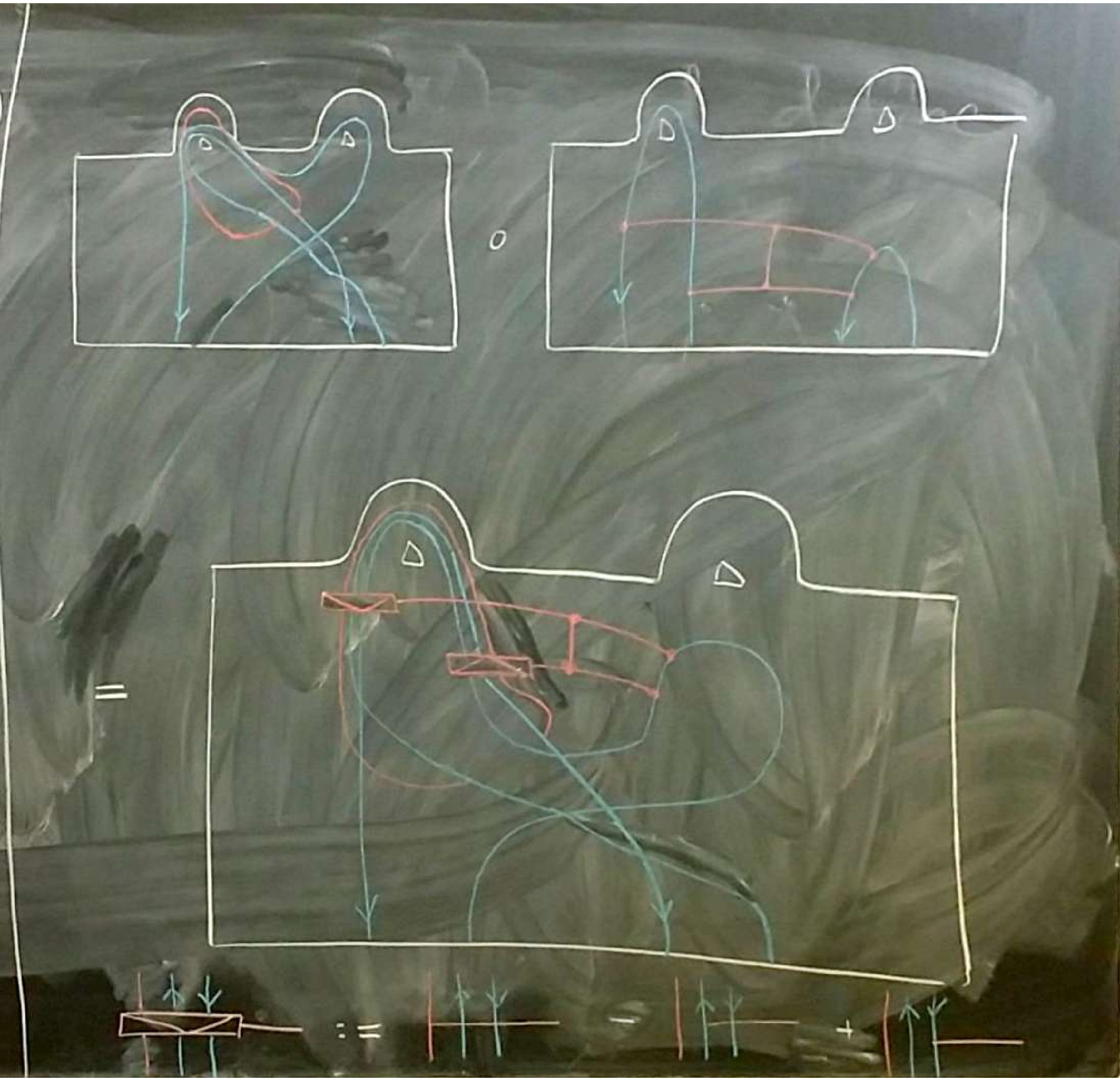
$[m=2, n=3]$

An (m,n) - Jacobi diag. is a homotopy class of

$(\Lambda_1 \ \Lambda_m) \cup D \longrightarrow V_m$

Def Let $\mathbb{A}$ be the (graded linear) cat whose objects are $m \geq 0$,

- morph. $m \to n$ are linear comb of (m,n)-Jacobi diag modulo (STU) 
- the $\circ$ law is a "discrete" version of the $\circ$ in $\mathbb{B}$



(III) - A generalization of the Kontsevich integral

Thm (with HABIRO '17)

For any Drinfeld associator φ, we have a functor

$$Z = Z^{\varphi} : \mathcal{B}_q \longrightarrow \hat{\mathbb{A}}$$

is filtered and $\mathrm{Gr}\, Z : \mathrm{Gr}^{\nabla} \mathbb{Q}\mathcal{B} \xrightarrow[\simeq]{} \mathbb{A}$

Ex

 $\in \mathcal{B}(4,3)$

A Drinfeld associator is a $\varphi \in \mathbb{Q}\langle\langle X, Y \rangle\rangle$ of the form

$$\varphi(X,Y) = \exp\left(\frac{1}{24}[X,Y] + \sum \text{iterated comm of deg} > 2\right)$$

which is a sol of the "pentagon" equation

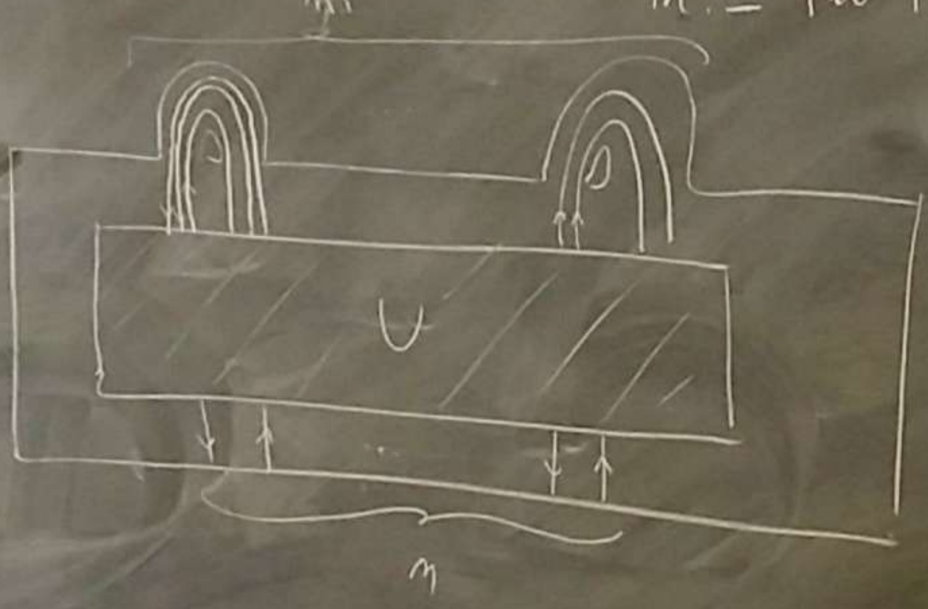
SKETCH OF THE CONSTRUCTION.

Let $T \in \mathcal{B}_q(v, w)$

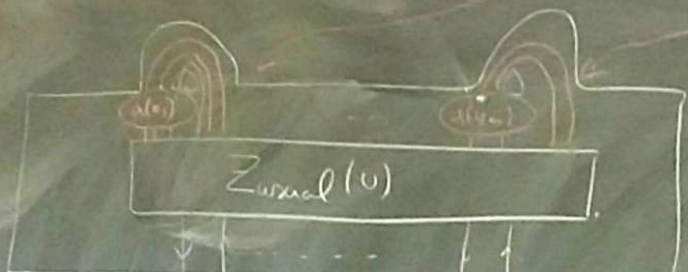
$m = |v|$

$n = |w|$

$T =$



leg from φ

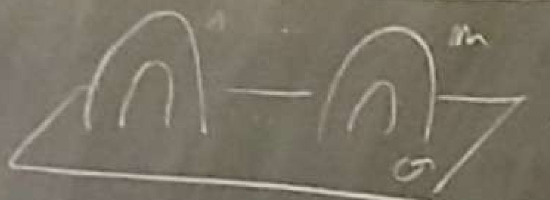
$Z(T) =$  $\in \hat{A}(m, n)$

N.B. $V_m =$  $\times [0,1]$

[Andersen – Mattes – Reshetikhin '98]

$\square$

(IV) Relationship with TQFT's

$\forall m \geq 0$, 

$= \partial_+ V_m$

Def (Crane & Yetter '99, Kerler '99)

Let Cob be the cat where
- objects are $m \geq 0$
- morphisms $m \to n$ are cobordisms $\Sigma_{m,1} \to \Sigma_{n,1}$
- $\circ$ is def by gluing

$$\mathcal{B}(m,m') \ni T \longmapsto V_m \setminus \operatorname{int} N(T)$$

$$
\begin{tikzcd}
\mathcal{B} \arrow[r, hook] \arrow[dr] & \mathcal{C}ob \\
 & \mathcal{L}\mathcal{C}ob \arrow[u, phantom, "\cup"]
\end{tikzcd}
$$

$\mathcal{L}\mathcal{C}ob$ ↳ Lagrangian

[with Cheptea & Habiro '07] The LMO invariant of homology 3-spheres can be extended to a functor $\mathcal{L}\mathcal{C}ob_q \xrightarrow{\tilde{Z}} {}^{ts}\mathcal{A}$

another cat of Jacobi diagrams

<u>Thm</u> (with HABIRO '17)

There exists a non-faithful functor

$$\hat{\mathbb{A}} \xrightarrow{\chi} {}^{\mathrm{to}}\!\mathcal{A}$$

such that

$$\begin{tikzcd}
 & \hat{\mathbb{A}} \arrow[dr, "\chi"] & \\
\mathcal{B}_q \arrow[ur, "Z"] \arrow[dr, hook] & & {}^{\mathrm{to}}\!\mathcal{A} \\
 & 2\mathcal{C}ob_q \arrow[ur, "\tilde{Z}"'] &
\end{tikzcd}$$

κ is a diag version of the Magnus expansion

$$\begin{cases} F(x_1, \dots, x_n) \longmapsto \mathbb{Q}\langle\langle X_1, \dots, X_n \rangle\rangle \\ x_i \longmapsto 1 + X_i + \dots = \exp(X_i) \end{cases}$$

κ is an analogue of the "hair map" in [Garoufalidis–Kricker '04]

subgroup of $MCG(\Sigma_{g,1}, \partial\Sigma_{g,1})$	functor	tree-level
Lagrangian subgroup (Torelli group)	$\tilde{Z}$	Johnson homomorphism
handlebody group	Z	equivariant Johnson homomorphisms